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Dimensional regularisation and factorisation schemes in the BFKL equation at subleading level

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Abstract: We study the anomalous dimensions and coefficient functions generated by the BFKL equation in $4+2\varepsilon$ dimensions, by investigating both running coupling effects, and the inclusion of the full next-to-leading kernel. After generalising the Fourier representation of the solutions to this case, we analyse the β-dependent renormalisation-group factorisation of anomalous dimension and coefficient contributions to the gluon density. We derive on this basis the normalisation factor of the Q_0-scheme with respect to the $\overline{\rm MS}$-scheme, including β-dependent corrections to it, and we outline the derivation of the full next-to-leading contributions. We also provide an expression for the resummed γ_{qg} in the $\overline{\rm MS}$-scheme which exhibits its universality and is explicit up to quadratures.

Keywords: NLO Computations, Deep Inelastic Scattering, QCD, Parton Model.

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1. Introduction

The relationship of the BFKL equation [1] — describing small-x evolution in QCD — with the DGLAP equation [2] — describing Q^2 evolution — has been the subject of several investigations in the nineties [3 – 9], in the attempt of providing a unified picture of small-x physics in QCD. It is well known that consistency of the BFKL approach with renormalisation group (RG) factorisation is achieved by means of resummation formulas of large contributions at small-x to the quark and gluon anomalous dimensions and coefficient functions, which have been derived at the leading-log x (LLx) level [3] and, in part, at the next-to-leading (NLx) one [5, 6].

In particular, Catani and Hautmann [6] have studied the LLx BFKL equation in $4+2\varepsilon$ dimensions with dimensional regularisation (and frozen coupling) and have derived, on this basis, the resummation of leading logarithms [5, 6] for the gluon coefficient function — usually called $R(\alpha_{\rm s})$ — and for the quark anomalous dimension $\gamma_{\sf qg}$ in the $\overline{\rm MS}$- and DIS-schemes. This calculation extends in part into the NLx level, not only because of $\gamma_{\sf qg}$ being next-to-leading, but also because of the R-factor, which provides NLx corrections to the gluon anomalous dimension $\gamma_{\sf gg}$ when running coupling effects are turned on [5, 10].

It is known, on the other hand, that full NLx terms [11, 12] are quite large and negative, and that doubly resummed approaches [13 − 15] are required in order to stabilise the subleading-log series. This raises the question of a better analysis of anomalous dimensions and coefficient functions at subleading level in various factorisation schemes. While the $\boldsymbol{k}$-factorisation schemes (like the Q_0-scheme [16], characterised by an off-shell initial parton) have been pushed to the doubly-resummed level [14], the minimal subtraction one (characterised by dimensional regularisation, and normally used in fixed order calculations) needs yet to be extended to a full treatment of NLx terms.

The purpose of the present paper is to devise a method to perform such analysis, and to work it out in detail in the case of the BFKL equation with running coupling, which already contains quite important subleading effects. The method is then extended to a full treatment of NLx coefficient terms, which require the ε-dependence of the NLx BFKL kernel at least up to $\mathcal{O}(\varepsilon)$. Unfortunately, the latter is yet to be extracted from the literature [7 − 9].

The main tool of our analysis is the generalisation to $4+2\varepsilon$ dimensions of the γ-representation of the gluon density — that is, of a Fourier representation of the BFKL solution in which γ is conjugate to $t \equiv \log(\boldsymbol{k}^2/\mu^2)$. While for $\varepsilon = 0$ the running-coupling equation is a differential equation in γ, for $\varepsilon \neq 0$ it becomes a finite-difference equation which is treated in sections 2 and 3 and in appendix A. This allows one to write the gluon density in a generalised Q_0-scheme[1] as the product of an anomalous dimension exponential and of a fluctuation factor that we call $\mathcal{N}$. We then show in section 2 that the factor $\mathfrak{R} \equiv R/\mathcal{N}$ is due to the $\mathcal{O}(\varepsilon)$ dependence of the LLx gluon anomalous dimension. This result offers an interpretation of the mismatch of R and $\mathcal{N}$ coefficients, and a hint to possible generalisations, investigated in sections 3–5.

The general case with running coupling ($b > 0$) is treated in section 3, and is qualitatively similar to the frozen coupling ($b = 0$) case, except that the beta-function

$$\beta(\bar{\alpha}_{\rm s}) \equiv \frac{{\rm d}\bar{\alpha}_{\rm s}(t)}{{\rm d}t} = \varepsilon\bar{\alpha}_{\rm s} - b\bar{\alpha}_{\rm s}^2 + \mathcal{O}\left(\bar{\alpha}_{\rm s}^3\right) \tag{1.1}$$

has both the dimensional contribution (ε-term) and the running-coupling one (b-term). We are thus able to explain how RG factorisation works for coefficient and anomalous dimension

[1] The label Q_0 referred originally [16] to the fact that the initial gluon, defined by $\boldsymbol{k}$-factorisation, was set off-mass-shell ($\boldsymbol{k}^2 = Q_0^2$) in order to cutoff the infrared singularities. It will be shown, however, that the effective anomalous dimension at scale $\boldsymbol{k}^2 \gg Q_0^2$ is independent of the cut-off procedure, whether of dimensional type or of off-mass-shell one.

parts for $b > 0$, thus relating the Q_0-scheme (with dimensional regularisation) and the $\overline{\text{MS}}$-scheme in an unambiguous way. In particular, we confirm that the two anomalous dimensions differ by the quantity $\mathrm{d}\log R/\mathrm{d}t$.

Sections 4 and 5 are devoted to the calculation of the NLx corrections to R. In section 4 we treat the corrections due to the running coupling, which require the (known) ε-dependence of the leading kernel eigenvalue up to $\mathcal{O}\left(\varepsilon^2\right)$, and the corresponding refinement of the γ-representation and of the saddle-point fluctuation formalism. In section 5 we calculate the remaining NLx corrections, due to the inclusion of the NLx kernel. Here the finite difference equation involves two steps, and is truncated at NLx level by an iterative procedure. The final result involves the $\mathcal{O}(\varepsilon)$ corrections to the NLx kernel eigenvalue, which are not yet explicitly known.

In section 6 we reconsider the calculation of γ_{qg} in the $\overline{\text{MS}}$-scheme. Contrary to the case of the DIS-scheme, in which the coefficient function C_{qg} is set to zero, no closed form resummation formula is yet available for $\gamma_{\mathsf{qg}}^{(\overline{\text{MS}})}$. Catani and Hautmann are able to provide a recursive method in order to disentangle C_{qg} from γ_{qg}, so as to calculate a number of terms of their expansion in $\bar{\alpha}_{\mathrm{s}}/\omega$, ω being the Mellin moment conjugate to x, a result further improved in the later literature [17]. Here we use the γ-representation of the gluon density in order to do the same calculation: this allows us to provide an explicit resummation formula which exhibits the universality of γ_{qg} and involves only quadratures of functions which are known in principle. However, the latter calculation requires an all-order ε-expansion of the coefficient function of the gluon. An interesting byproduct of the universality analysis is the proof (appendix B) that the *off-shell* P_{qg} splitting function introduced in [6] is indeed process-independent.

In the final section 7 we summarise our results and we discuss their relevance for the improved resummations of splitting functions [14, 15].

2. γ-representation and R-factor in the LLx case

Investigating the BFKL equation in $4+2\varepsilon$ dimensions is of importance on its own, because — in addition to the ultraviolet (UV) role of dimensional regularisation — a positive ε parameter allows to regularise the mass-shell singularities and, in some regime (see section 3) it avoids the Landau pole. For this reason we shall consider $\varepsilon > 0$ as an infrared cutoff, in alternative to setting off-mass-shell the initial condition for partonic evolution, as done in the Q_0-scheme [16]. Our purpose is, at large, to understand the relationship of such two kinds of schemes, whether or not a minimal subtraction is assumed in the partonic densities.

Our procedure involves several steps. The first step consists in computing the BFKL gluon density g_{ε,Q_0} with an off-mass-shell cutoff Q_0. Next, one performs the limit of vanishing cutoff $Q_0 \to 0$, obtaining an ε-singular gluon density g_ε. Then, one factorises the collinear singularities of g_ε according to some scheme choice (say $\overline{\text{MS}}$); this leads to a different gluon density $g^{(\overline{\text{MS}})}$. Finally, one determines the relation between the two densities in the form $g_\varepsilon = R \cdot g^{(\overline{\text{MS}})}$, where the gluon coefficient $R(\alpha_{\mathrm{s}}, \omega)$ is finite in the $\varepsilon \to 0$ limit and can be calculated in the LLx approximation (in this case it only depends on the ratio

α_s/ω) and also in the NLx approximation by computing the $\mathcal{O}(\alpha_s)$ corrections. In this section we start with the LLx calculation of R, in which we recover the results of [5, 6] by introducing a new method based on a Fourier representation of the BFKL gluon density g_ε. This method is particularly suitable for determining the anomalous dimensions and will be generalised to NLx level in the following sections.

Since the gauge coupling g is dimensionful in $4+2\varepsilon$ dimensions, we shall introduce, as usual, the renormalisation scale μ, the dimensionless coupling in the $\overline{\mathrm{MS}}$-scheme

$$\alpha_s \equiv \frac{(g\mu^\varepsilon)^2}{(4\pi)^{1+\varepsilon} e^{\varepsilon\psi(1)}}\,, \qquad \bar{\alpha}_s \equiv \alpha_s \frac{N_c}{\pi}\,, \tag{2.1}$$

and the parameter $t \equiv \log(\boldsymbol{k}^2/\mu^2)$ in terms of which the running coupling is $\bar{\alpha}_s(t) \equiv \bar{\alpha}_s e^{\varepsilon t}$. Therefore, the LL$x$ equation shows a running coupling with infrared free evolution corresponding to the dimensional contribution of the beta function (1.1) in the $b \to 0$ limit. Since $\bar{\alpha}_s(t) \to 0$ for $t \to -\infty$, we can set on-shell ($\boldsymbol{k}_0 = 0$) the initial condition for the gluon Green's function, so that ε acquires the role of infrared cutoff. The ensuing solution of the LLx BFKL equation is naturally expressed as a power series in $\bar{\alpha}_s(t)$, which was extensively studied in [6].

Our purpose is to recast the solution for the gluon density mentioned above in the γ-representation form — γ being conjugate to t — so as to be able to describe in a simpler way its anomalous dimension behaviour and to rederive in this way the LLx expression for the R factor. The LLx BFKL equation for the unintegrated gluon density $\mathcal{F}$ in $4+2\varepsilon$ dimensions reads (the ω-dependence of $\mathcal{F}$ is understood)

$$\begin{aligned}\mathcal{F}_\varepsilon(\boldsymbol{k}) &= \delta^{(2+2\varepsilon)}(\boldsymbol{k}) + \frac{1}{\omega}\int \frac{d^{2+2\varepsilon}\boldsymbol{k}'}{(2\pi)^{2+2\varepsilon}}\, \mathcal{K}_\varepsilon(\boldsymbol{k},\boldsymbol{k}')\mathcal{F}_\varepsilon(\boldsymbol{k}') \\ &= \delta^{(2+2\varepsilon)}(\boldsymbol{k}) + \frac{e^{\varepsilon\psi(1)}}{(\pi\boldsymbol{k}^2)^{1+\varepsilon}}\widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k})\,,\end{aligned} \tag{2.2}$$

$$\mathcal{K}_\varepsilon(\boldsymbol{k},\boldsymbol{k}') = g^2 N_c\left[\frac{1}{\pi(\boldsymbol{k}-\boldsymbol{k}')^2} - (\pi\boldsymbol{k}^2)^\varepsilon \frac{\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)}{\varepsilon\Gamma(1+2\varepsilon)}\delta^{2+2\varepsilon}(\boldsymbol{k}-\boldsymbol{k}')\right] \tag{2.3}$$

and its power series solution, derived in [6], is

$$\widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k}) = \frac{\bar{\alpha}_s(t)}{\omega}\left[1 + \sum_{n=1}^{\infty}\left(\frac{\bar{\alpha}_s(t)}{\omega}\right)^n \prod_{m=1}^{n}\chi_\varepsilon(m\varepsilon)\right], \tag{2.4}$$

where $\bar{\alpha}_s(t) = \bar{\alpha}_s e^{\varepsilon t}$ and

$$\chi_\varepsilon(\gamma) = \frac{e^{\varepsilon\psi(1)}\Gamma(1+\varepsilon)}{\varepsilon}\left[\frac{\Gamma(\gamma)\Gamma(1-\gamma)}{\Gamma(\gamma+\varepsilon)\Gamma(1-\gamma+\varepsilon)} - \frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1+2\varepsilon)}\right] \tag{2.5}$$

is the LLx "characteristic function" of $\mathcal{K}_\varepsilon$ in $4+2\varepsilon$ dimensions, defined by

$$\int \frac{d^{2+2\varepsilon}\boldsymbol{k}'}{(2\pi)^{2+2\varepsilon}}\, \mathcal{K}_\varepsilon(\boldsymbol{k},\boldsymbol{k}')(\boldsymbol{k}'^2)^{\gamma-1-\varepsilon} \equiv \bar{\alpha}_s\chi_\varepsilon(\gamma)\frac{(\boldsymbol{k}^2)^{\gamma-1}}{\mu^{2\varepsilon}}\,. \tag{2.6}$$

Eq. (2.4) is obtained by an iterative procedure, where the application of $\mathcal{K}_\varepsilon$ to the δ-function yields a term proportional to $\bar{\alpha}_s(\boldsymbol{k}^2/\mu^2)^\varepsilon = \bar{\alpha}_s(t)$, and any further iteration yields an additional factor of $\bar{\alpha}_s(t)$ times the corresponding characteristic function, whose normalisation is slightly different from that of ref. [6].

The resulting series (2.4) is well behaved in the infrared ($t \to -\infty$) apart from the $1/\varepsilon$ poles, but is not really suitable in the ultraviolet ($t \to +\infty$), where the variable $\bar{\alpha}_s(t)/\omega$ grows, and at most a finite radius of convergence is expected. We thus look for the solution of eq. (2.2) in the γ-representation form

$$\widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k}) = \int_{\Re\gamma=c} \frac{\mathrm{d}\gamma}{2\pi\mathrm{i}}\, \mathrm{e}^{\gamma t} f_\varepsilon(\gamma)\,, \tag{2.7}$$

where the Fourier variable γ has the interpretation of a continuum in which $\gamma = n\varepsilon$ is the lattice counterpart. We expect eq. (2.7) to be best suited for anomalous dimension properties in which $n \to \infty$ and $\varepsilon \to 0$ with $\gamma = n\varepsilon$ kept fixed.

Let us start noticing that the "ansatz" (2.7), when replaced in the BFKL equation with a general initial condition $f_\varepsilon^{(0)}(\gamma)$, leads to a finite difference equation of the form

$$f_\varepsilon(\gamma+\varepsilon) = f_\varepsilon^{(0)}(\gamma+\varepsilon) + \frac{\bar{\alpha}_s}{\omega}\chi_\varepsilon(\gamma) f_\varepsilon(\gamma)\,. \tag{2.8}$$

In the following we shall often consider the case

$$f_\varepsilon^{(0)}(\gamma+\varepsilon) = f^{(0)}(\gamma) \equiv \frac{\bar{\alpha}_s}{\omega}\frac{\mathrm{e}^{\gamma T}}{\gamma}\,, \tag{2.9}$$

which corresponds to the initial condition

$$\widetilde{\mathcal{F}}_\varepsilon^{(0)}(\boldsymbol{k}) = \frac{\bar{\alpha}_s \mathrm{e}^{\varepsilon t}}{\omega}\Theta(t+T)\,, \tag{2.10}$$

i.e. to the one expected from eq. (2.4) with a cutoff at $\boldsymbol{k}^2 = Q_0^2 \equiv \mu^2\mathrm{e}^{-T}$. In this way we can study the role of the cutoff, by solving (2.8) with T fixed, and performing the limits $T \to +\infty$, $\varepsilon \to 0$ either in this order, or in reverse order.

We shall solve eq. (2.8) in two steps, starting from the homogeneous equation

$$h_\varepsilon(\gamma+\varepsilon) = \frac{\bar{\alpha}_s}{\omega}\chi_\varepsilon(\gamma) h_\varepsilon(\gamma) \equiv \mathrm{e}^{L_\varepsilon(\gamma)} h_\varepsilon(\gamma)\,, \tag{2.11}$$

where we take the ansatz

$$h_\varepsilon(\gamma) = \exp\left\{\int^\gamma \mathcal{L}_\varepsilon(\gamma')\frac{\mathrm{d}\gamma'}{\varepsilon}\right\} \equiv \exp\left\{S_\varepsilon(\gamma)\right\} \tag{2.12}$$

so that eq. (2.11) implies

$$S_\varepsilon(\gamma+\varepsilon) - S_\varepsilon(\gamma) = \int_\gamma^{\gamma+\varepsilon} \mathcal{L}_\varepsilon(\gamma')\frac{\mathrm{d}\gamma'}{\varepsilon} = L_\varepsilon(\gamma)\,. \tag{2.13}$$

We show in appendix A.1 that the solution for the "lagrangian" $\mathcal{L}_\varepsilon(\gamma)$ is expressed, under appropriate conditions, in terms of the Bernoulli numbers B_n as follows

$$\mathcal{L}_\varepsilon(\gamma) = \sum_{n=0}^{\infty} \frac{B_n}{n!}\varepsilon^n L_\varepsilon^{(n)}(\gamma)\,, \tag{2.14}$$

where $L^{(n)}$ is the n-th derivative of L with respect to γ, and the generating function is

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{\mathrm{e}^z - 1}\,, \tag{2.15}$$

so that

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \cdots . \tag{2.16}$$

Correspondingly the "action" takes the form

$$S_\varepsilon(\gamma) = \frac{1}{\varepsilon}\int^\gamma L_\varepsilon(\gamma')\mathrm{d}\gamma' - \frac{1}{2}L_\varepsilon(\gamma) + \frac{\varepsilon}{12}L'_\varepsilon(\gamma) + \mathcal{O}\left(\varepsilon^2\right)\,, \tag{2.17}$$

where L_ε can be further expanded in ε as follows

$$L_\varepsilon(\gamma) \equiv \log\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\chi_\varepsilon(\gamma)\right) = \log\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\chi_0(\gamma)\right) + \varepsilon\frac{\chi_1(\gamma)}{\chi_0(\gamma)} + \varepsilon^2\left(\frac{\chi_2(\gamma)}{\chi_0(\gamma)} - \frac{1}{2}\frac{\chi_1^2(\gamma)}{\chi_0^2(\gamma)}\right) + \mathcal{O}\left(\varepsilon^3\right) \tag{2.18}$$

and the BFKL eigenvalue function

$$\chi_\varepsilon = \chi_0 + \varepsilon\chi_1 + \varepsilon^2\chi_2 + \mathcal{O}\left(\varepsilon^3\right) \tag{2.19}$$

is given, according to eq. (2.5), by

$$\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \tag{2.20}$$

$$\chi_1(\gamma) = \frac{1}{2}\left[\chi_0^2(\gamma) + 2\psi'(1) - \psi'(\gamma) - \psi'(1-\gamma)\right] \tag{2.21}$$

$$\chi_2(\gamma) = \chi_0(\gamma)\left[\chi_1(\gamma) - \frac{1}{2}\psi'(1)\right] - \frac{1}{3}\chi_0^3(\gamma) + \frac{1}{6}\left[8\psi''(1) - \psi''(\gamma) - \psi''(1-\gamma)\right]. \tag{2.22}$$

Therefore, the solution of the homogeneous equation in t-space (eq. (2.2) without delta term) takes the form

$$\widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k}) = \int \mathrm{d}\gamma\, \frac{\mathrm{e}^{\gamma t}}{\sqrt{\chi_0(\gamma)}} \exp\left\{\frac{1}{\varepsilon}\int^\gamma \log\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\chi_0(\gamma')\right)\mathrm{d}\gamma' + \int^\gamma \frac{\chi_1(\gamma')}{\chi_0(\gamma')}\mathrm{d}\gamma'\right\} \times \left[1 + \mathcal{O}\left(\varepsilon\right)\right], \tag{2.23}$$

where we have truncated the ε-expansion of the exponent up to the finite terms. The choice of the lower bound of the γ' integrals is of no concern, since the normalisation of the solution of the homogeneous equation is arbitrary.

The inhomogeneous equation (2.8) has, on the other hand, the iterative solution

$$\begin{aligned} f_\varepsilon(\gamma) &= f^{(0)}(\gamma-\varepsilon) + f^{(0)}(\gamma-2\varepsilon)\frac{\bar{\alpha}_\mathrm{s}}{\omega}\chi_\varepsilon(\gamma-\varepsilon) + f^{(0)}(\gamma-3\varepsilon)\frac{\bar{\alpha}_\mathrm{s}}{\omega}\chi_\varepsilon(\gamma-2\varepsilon)\frac{\bar{\alpha}_\mathrm{s}}{\omega}\chi_\varepsilon(\gamma-\varepsilon) + \cdots \\ &= \sum_{n=1}^{\infty} f^{(0)}(\gamma-n\varepsilon)\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)^{n-1}\prod_{m=1}^{n-1}\chi_\varepsilon(\gamma-m\varepsilon) = \sum_{n=1}^{\infty} f^{(0)}(\gamma-n\varepsilon)\frac{h_\varepsilon(\gamma)}{h_\varepsilon(\gamma+\varepsilon-n\varepsilon)}\,, \end{aligned} \tag{2.24}$$

which reduces to the power series solution (2.4) in the $T \to +\infty$ limit, as analysed in more detail in appendix A.2. Here we just note the $\varepsilon = 0$ limit of eq. (2.24) at fixed T. Due to the fact that $h_\varepsilon(\gamma)$ satisfies eq. (2.11), the ratios of h's has a non trivial $\varepsilon \to 0$ limit, and we obtain

$$f_0(\gamma) = f^{(0)}(\gamma) \sum_{n=0}^{\infty} \mathrm{e}^{nL_0(\gamma)} = \frac{f^{(0)}(\gamma)}{1 - \frac{\bar{\alpha}_\mathrm{s}}{\omega}\chi_0(\gamma)}\,, \tag{2.25}$$

as expected for the solution of the BFKL equation with frozen coupling and with cutoff. We get in fact, up to higher twist corrections,

$$\widetilde{\mathcal{F}}_0(\boldsymbol{k}) = \frac{\mathrm{e}^{\gamma_0 T}}{-\gamma_0\, \chi_0'(\gamma_0)} \left(\frac{\boldsymbol{k}^2}{\mu^2}\right)^{\gamma_0} \Theta(\boldsymbol{k}^2 - \mu^2 \mathrm{e}^{-T})\,, \tag{2.26}$$

where $\gamma_0(\bar{\alpha}_\mathrm{s}/\omega)$ is the LLx anomalous dimension, defined by

$$1 = \frac{\bar{\alpha}_\mathrm{s}}{\omega}\chi_0(\gamma_0)\,, \qquad \gamma_0\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) = \frac{\bar{\alpha}_\mathrm{s}}{\omega} + \mathcal{O}\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)^4\,. \tag{2.27}$$

The result (2.26) shows the customary infrared singular T-dependence of the Q_0-scheme, determined by γ_0.

Our emphasis in this paper is, however, in the limit of (2.24) for $\varepsilon \ll 1$ and $T \to +\infty$, in which the solution (2.4) is recovered, and the sum in the r.h.s. of (2.24) can be replaced by an integral in the "continuum" variable $\gamma' = \gamma - m\varepsilon$, apart from some ε-dependent corrections. In order to specify such corrections, we rewrite eq. (2.8) in terms of $\rho_\varepsilon(\gamma) \equiv f_\varepsilon(\gamma)/h_\varepsilon(\gamma)$ in the simpler form

$$\rho(\gamma + \varepsilon) - \rho(\gamma) = \frac{f^{(0)}(\gamma)}{h_\varepsilon(\gamma + \varepsilon)} = \frac{f^{(0)}(\gamma)}{h_\varepsilon(\gamma)} \mathrm{e}^{-L_\varepsilon(\gamma)}\,, \tag{2.28}$$

which is of the type (2.13) and is solved in terms of Bernoulli numbers as in (2.14). The r.h.s. of (2.28) has, however, $1/\varepsilon$ singularities in the exponent (cf. eqs. (2.12) and (2.17)), and its derivatives generate eventually a non trivial correction factor (see appendix A.3) having a finite $\varepsilon \to 0$ limit, as follows:

$$\rho_\varepsilon(\gamma) = \int^{\gamma} \frac{\mathrm{d}\gamma'}{\varepsilon} \frac{f^{(0)}(\gamma')}{h_\varepsilon(\gamma' + \varepsilon)} \frac{L_0(\gamma') - \varepsilon T}{1 - \mathrm{e}^{-L_0(\gamma') + \varepsilon T}} \times \left[1 + \mathcal{O}(\varepsilon)\right]\,, \tag{2.29}$$

where we have kept terms $\sim \varepsilon T$, that we consider a number of order unity.

Finally, by replacing eq. (2.29) into (2.7) we get the solution in the continuum limit

$$\begin{aligned} \widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k}) = \int \frac{\mathrm{d}\gamma}{2\pi\mathrm{i}} \int^{\gamma'} \mathrm{d}\gamma' \; & \frac{\exp\left\{\gamma t + \frac{1}{\varepsilon}\int_0^\gamma L_0(z)\mathrm{d}z + \gamma' T - \frac{1}{\varepsilon}\int_0^{\gamma'} L_0(z)\mathrm{d}z + \int_{\gamma'}^{\gamma} \frac{\chi_1(z)}{\chi_0(z)}\mathrm{d}z\right\}}{\varepsilon\gamma'\sqrt{\chi_0(\gamma)}\sqrt{\chi_0(\gamma')}} \times \\ & \times \frac{L_0(\gamma') - \varepsilon T}{1 - \mathrm{e}^{-L_0(\gamma') + \varepsilon T}}\,, \end{aligned} \tag{2.30}$$

where we have truncated the ε-expansion to the finite terms. This solution has the form of a double γ-representation, similarly to the customary case $b \neq 0$, $\varepsilon = 0$, in which the ε-parameter replaces b by providing an infrared free running coupling.

Our goal is now to investigate the $T \to +\infty$ limit of eq. (2.30) by removing the artificial infrared cutoff $Q_0^2 = \mu^2 \mathrm{e}^{-T}$. In this limit the leading γ'-dependent phase is generally large and given by

$$E(\gamma') = \gamma' T - \frac{1}{\varepsilon}\int_0^{\gamma'} L_0(z)\mathrm{d}z\,, \tag{2.31}$$

so that the γ'-integral is dominated by the saddle point $\bar{\gamma}'$:

$$E'(\bar{\gamma}') = T - \frac{1}{\varepsilon} L_0(\bar{\gamma}') = 0\,, \tag{2.32}$$

which implies

$$1 = \frac{\bar{\alpha}_{\rm s}(-T)}{\omega} \chi_0(\bar{\gamma}')\,. \tag{2.33}$$

Therefore, $\bar{\gamma}' \sim \bar{\alpha}_{\rm s}(-T)/\omega \to 0$ for $T \to +\infty$[2] and, by taking into account also the γ'-fluctuations

$$\sigma_{\gamma'} \equiv \frac{1}{\sqrt{E''(\bar{\gamma}')}} = \sqrt{\frac{\varepsilon \chi_0(\bar{\gamma}')}{-\chi_0'(\bar{\gamma}')}}\,, \tag{2.34}$$

all T-dependent factors in (2.30) cancel out, yielding

$$\widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k}) \overset{T\to+\infty}{\longrightarrow} \int \frac{\mathrm{d}\gamma}{\sqrt{2\pi\varepsilon}} \frac{1}{\sqrt{\chi_0(\gamma)}} \exp\left\{\gamma t + \frac{1}{\varepsilon}\int_0^\gamma L_0(\gamma')\mathrm{d}\gamma' + \int_0^\gamma \frac{\chi_1(\gamma')}{\chi_0(\gamma')}\mathrm{d}\gamma' + \mathcal{O}(\varepsilon)\right\}\,, \tag{2.35}$$

which is just a solution (2.23) of the *homogeneous* equation with an appropriate normalisation.

Eq. (2.35) is the main result of this section and shows the mechanism by which the solution becomes independent of the details of the initial condition in the $T \to +\infty$ ($Q_0 \to 0$) limit. In fact, due to the infinite evolution path from $-\infty$ to t, the shape of the solution reduces to the one of the homogeneous equation, and the initial condition determines only the normalisation. This already suggests the factorisation of the $1/\varepsilon$ singularities which have replaced the T-dependence in eq. (2.35). In order to prove such factorisation we resort again to the saddle point method in order to evaluate the $\varepsilon \to 0$ behaviour of (2.35). The stationarity condition is now

$$\varepsilon t + L_0(\bar{\gamma}) = 0 \qquad \Longleftrightarrow \qquad 1 = \frac{\bar{\alpha}_{\rm s}(t)}{\omega}\chi_0(\bar{\gamma})\,, \tag{2.36}$$

with a stable fluctuation along the real axis for $\chi_0'(\bar{\gamma}) < 0$, thus yielding the result ($\bar{\gamma} \equiv \bar{\gamma}_t = \gamma_0\big(\bar{\alpha}_{\rm s}(t)/\omega\big)$)

$$\widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k}) = \frac{1}{\sqrt{-\chi_0'(\bar{\gamma}_t)}} \exp\left\{\bar{\gamma}_t\, t + \frac{1}{\varepsilon}\int_0^{\bar{\gamma}_t} L_0(\gamma')\mathrm{d}\gamma' + \int_0^{\bar{\gamma}_t} \frac{\chi_1(\gamma')}{\chi_0(\gamma')}\mathrm{d}\gamma'\right\} \times \big[1 + \mathcal{O}(\varepsilon)\big]\,. \tag{2.37}$$

Finally, we calculate the integrated gluon density in the form

$$\begin{aligned} g_\varepsilon(t) &\equiv \int \mathrm{d}^{2+2\varepsilon}\boldsymbol{k}'\, \mathcal{F}_\varepsilon(\boldsymbol{k}')\,\Theta(\boldsymbol{k}^2 - \boldsymbol{k}'^2) \\ &= \frac{1}{\bar{\gamma}_t\sqrt{-\chi_0'(\bar{\gamma}_t)}} \exp\left\{\bar{\gamma}_t\, t + \frac{1}{\varepsilon}\int_0^{\bar{\gamma}_t} L_0(\gamma')\mathrm{d}\gamma' + \int_0^{\bar{\gamma}_t} \frac{\chi_1(\gamma')}{\chi_0(\gamma')}\mathrm{d}\gamma'\right\} \times \big[1 + \mathcal{O}(\varepsilon)\big], \end{aligned} \tag{2.38}$$

where we have performed an integration by parts of $\widetilde{\mathcal{F}}$ in (2.37).

[2] The solution with $\bar{\gamma}' \simeq 1$ is relevant in the infrared region $\boldsymbol{k}^2 \ll Q_0^2$ ($t \ll -T$), where it behaves as $\sim \boldsymbol{k}^2/Q_0^2 = \mathrm{e}^{T+t}$.

Some remarks are in order. Firstly, the saddle point condition (2.36) provides a real positive anomalous dimension only in the perturbative regime in which $\bar{\alpha}_s(t) < \omega/\chi_0(\frac{1}{2})$, value at which the LL$x$ anomalous dimension has the well known singularity at $\gamma = \frac{1}{2}$. Since, however, $\bar{\alpha}_s(t)$ increases with t for $\varepsilon > 0$, for large t the solutions to (2.36) will become necessary complex conjugate with $\Re\bar{\gamma} = \frac{1}{2}$, and $g_\varepsilon(t)$ will oscillate asymptotically. This shows how the perturbative series (2.4) behaves beyond its convergence radius, so that $\mathcal{F}(\boldsymbol{k})$ stays marginally square-integrable,[3] as required by the γ-representation.

Secondly, eq. (2.38) shows, in the perturbative regime, a fluctuation factor $\mathcal{N}$ and an anomalous dimension exponential, in the form

$$g_\varepsilon(t) = \mathcal{N}\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) \exp\left\{\int_{-\infty}^{t} \mathrm{d}\tau \, \left[\gamma_0\Big(\frac{\bar{\alpha}_s(\tau)}{\omega}\Big) + \varepsilon\gamma_1\Big(\frac{\bar{\alpha}_s(\tau)}{\omega}\Big)\right]\right\} \times \left[1 + \mathcal{O}(\varepsilon)\right] \tag{2.39}$$

$$\mathcal{N}\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) \equiv \frac{1}{\bar{\gamma}_t\sqrt{-\chi_0'(\bar{\gamma}_t)}}\,, \tag{2.40}$$

where

$$\gamma_1\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) \equiv -\frac{\chi_1(\bar{\gamma}_t)}{\chi_0'(\bar{\gamma}_t)} \tag{2.41}$$

is the $\mathcal{O}(\varepsilon)$ correction to the BFKL anomalous dimension. This interpretation follows simply by identifying the exponents in (2.39) and (2.38), which have the same t-derivative (by the saddle-point condition (2.36)) and the same value at $t \to -\infty$ ($\bar{\gamma}_t = 0$). Note that the $1/\varepsilon$ singularity of the exponent is a consequence of the boundary at $t \to -\infty$ in (2.39), because of the jacobians

$$\mathrm{d}t = \frac{\mathrm{d}\bar{\alpha}_s}{\varepsilon\bar{\alpha}_s} = -\frac{\chi_0'(\bar{\gamma}_t)}{\varepsilon\chi_0(\bar{\gamma}_t)}\mathrm{d}\bar{\gamma}_t\,, \tag{2.42}$$

which relate the various forms of the $1/\varepsilon$ exponent:

$$\int_{-\infty}^{t} \mathrm{d}\tau \; \gamma_0\Big(\frac{\bar{\alpha}_s(\tau)}{\omega}\Big) = \frac{1}{\varepsilon}\int_0^{\bar{\alpha}_s(t)} \frac{\mathrm{d}\alpha}{\alpha}\;\gamma_0\Big(\frac{\alpha}{\omega}\Big) = \bar{\gamma}_t\, t + \frac{1}{\varepsilon}\int_0^{\bar{\gamma}_t} \mathrm{d}\gamma'\; L_0(\gamma')\,. \tag{2.43}$$

We are now in a position to factorise the t-dependence from the $1/\varepsilon$ singularities in eq. (2.39). By simply subdividing the τ integration interval into $]-\infty, 0] \cup [0, t]$, we obtain[4]

$$g_\varepsilon(t) = \mathcal{N}\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) \exp\left\{\int_0^t \mathrm{d}\tau\;\gamma_0\Big(\frac{\bar{\alpha}_s(\tau)}{\omega}\Big)\right\} \mathfrak{R}\Big(\frac{\bar{\alpha}_s(0)}{\omega}\Big) \exp\left\{\frac{1}{\varepsilon}\int_0^{\bar{\alpha}_s(0)} \frac{\mathrm{d}\alpha}{\alpha}\;\gamma_0\Big(\frac{\alpha}{\omega}\Big)\right\}, \tag{2.44}$$

where we have defined the reduced coefficient factor

$$\mathfrak{R}(a) \equiv \frac{R(a)}{\mathcal{N}(a)} \equiv \exp\left\{\int_0^{\gamma_0(a)} \mathrm{d}\gamma' \; \frac{\chi_1(\gamma')}{\chi_0(\gamma')}\right\}\,, \qquad \Big(a \equiv \frac{\bar{\alpha}_s}{\omega}\Big)\,, \tag{2.45}$$

[3]That is, $\mathcal{F}(\boldsymbol{k}) \sim (\boldsymbol{k}^2)^{-1/2} \times$ oscillating function.

[4]Splitting the integration interval at $\tau = 0$ corresponds to choose the factorisation scale $\mu_f = \mu$, i.e., $t_f = 0$.

which arises because of the anomalous dimension correction $\varepsilon\gamma_1$ cancelling the $1/\varepsilon$ singularity of the jacobian (2.42) according to the identity

$$\varepsilon \int_{-\infty}^{0} \mathrm{d}\tau\, \gamma_1\Big(\frac{\bar{\alpha}_\mathrm{s}(\tau)}{\omega}\Big) = \int_0^{\bar{\alpha}_\mathrm{s}(0)} \frac{\mathrm{d}\alpha}{\alpha}\, \gamma_1\Big(\frac{\alpha}{\omega}\Big) = \int_0^{\gamma_0\left(\frac{\bar{\alpha}_\mathrm{s}(0)}{\omega}\right)} \mathrm{d}\gamma'\, \frac{\chi_1(\gamma')}{\chi_0(\gamma')}\,. \tag{2.46}$$

The expression (2.45) agrees, by eq. (2.21), with Eqs. (3.17,B.18) of Ref. [6]. Finally, the finite t-evolution in eq. (2.44) becomes simply

$$\mathcal{N}\Big(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\Big)\left(\frac{\boldsymbol{k}^2}{\mu^2}\right)^{\gamma_0\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)} \tag{2.47}$$

in the $\varepsilon \to 0$ limit, thus recovering the result [6]

$$g_\varepsilon(t) \to R\Big(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\Big)\left(\frac{\boldsymbol{k}^2}{\mu^2}\right)^{\gamma_0\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)} \exp\left\{\frac{1}{\varepsilon}\int_0^{\bar{\alpha}_\mathrm{s}} \frac{\mathrm{d}\alpha}{\alpha}\, \gamma_0\Big(\frac{\alpha}{\omega}\Big)\right\}, \tag{2.48}$$

where

$$\begin{aligned} R\Big(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\Big) &\equiv \mathcal{N}\Big(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\Big)\mathfrak{R}\Big(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\Big) \\ &= \left\{\frac{\Gamma(1-\gamma)\chi_0(\gamma)}{\Gamma(1+\gamma)[-\gamma\chi_0'(\gamma)]}\right\}^{1/2} \exp\left\{\gamma\,\psi(1) + \int_0^{\gamma} \mathrm{d}\gamma'\, \frac{\psi'(1)-\psi'(1-\gamma')}{\chi_0(\gamma')}\right\}\Bigg|_{\gamma=\gamma_0\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)}. \end{aligned} \tag{2.49}$$

This derivation shows the mechanism by which the R coefficient factor is obtained as the product of $\mathcal{N}$ (arising from the γ-fluctuations) and $\mathfrak{R}$ (arising from the ε-dependence of the BFKL anomalous dimension). In the following we shall generalise this mechanism to $b > 0$ and to further terms in the ε-expansion.

3. The running coupling equation and its factorisation properties

Our first purpose is to generalise to $4 + 2\varepsilon$ dimensions the BFKL equation with running coupling. Because of the ε-dependence of the β-function in eq. (1.1), the running coupling $\bar{\alpha}_\mathrm{s}(t)$ has the form

$$\frac{1}{\bar{\alpha}_\mathrm{s}(t)} - \frac{b}{\varepsilon} = \mathrm{e}^{-\varepsilon t}\left(\frac{1}{\bar{\alpha}_\mathrm{s}} - \frac{b}{\varepsilon}\right), \qquad \text{or} \qquad \bar{\alpha}_\mathrm{s}(t) = \frac{\bar{\alpha}_\mathrm{s}\mathrm{e}^{\varepsilon t}}{1 + b\bar{\alpha}_\mathrm{s}\frac{\mathrm{e}^{\varepsilon t}-1}{\varepsilon}}, \tag{3.1}$$

where $b = 11/12$ in the $N_f = 0$ limit. Due to the UV fixed point of eq. (1.1) at $\bar{\alpha}_\mathrm{s} = \varepsilon/b$, eq. (3.1) shows two distinct regimes, according to whether (i) $\bar{\alpha}_\mathrm{s} < \varepsilon/b$ or (ii) $\bar{\alpha}_\mathrm{s} > \varepsilon/b$. In the regime (i) $\bar{\alpha}_\mathrm{s}(t)$ runs monotonically from $\bar{\alpha}_\mathrm{s} = 0$ to $\bar{\alpha}_\mathrm{s} = \varepsilon/b$ for $-\infty < t < +\infty$, while in the regime (ii) $\bar{\alpha}_\mathrm{s}(t)$, starting from ε/b in the UV limit, goes through the Landau pole at $t_\Lambda = \log(1 - \varepsilon/b\bar{\alpha}_\mathrm{s})/\varepsilon < 0$, and reaches $\bar{\alpha}_\mathrm{s} = \varepsilon/b$ from below in the IR limit.

Since the LLx kernel $\mathcal{K}_\varepsilon(t,t')\mathrm{d}t'$ scales like $\mathrm{e}^{\varepsilon t}$, an equation which realises such $\bar{\alpha}_\mathrm{s}(t)$-evolution (and regimes) is obtained by setting

$$\mathcal{F}_{\varepsilon,b}(\boldsymbol{k}) = \delta^{(2+2\varepsilon)}(\boldsymbol{k}) + \frac{1}{\omega}\frac{1}{1 + b\bar{\alpha}_\mathrm{s}\frac{\mathrm{e}^{\varepsilon t}-1}{\varepsilon}} \int \frac{\mathrm{d}^{2+2\varepsilon}\boldsymbol{k}'}{(2\pi)^{2+2\varepsilon}}\, \mathcal{K}_\varepsilon(\boldsymbol{k},\boldsymbol{k}')\mathcal{F}_{\varepsilon,b}(\boldsymbol{k}')$$

$$= \delta^{(2+2\varepsilon)}(\boldsymbol{k}) + \frac{e^{\varepsilon\psi(1)}}{(\pi\boldsymbol{k}^2)^{1+\varepsilon}}\widetilde{\mathcal{F}}_{\varepsilon,b}(\boldsymbol{k})\,. \tag{3.2}$$

It is soon realised that $\widetilde{\mathcal{F}}_{\varepsilon,b}(\boldsymbol{k})$, given by (cf. eq. (A.5))

$$\widetilde{\mathcal{F}}_{\varepsilon,b}(\boldsymbol{k}) = \frac{1}{\omega}\frac{\bar{\alpha}_s e^{\varepsilon t}}{1+b\bar{\alpha}_s\frac{e^{\varepsilon t}-1}{\varepsilon}}\left[1+K_\varepsilon\widetilde{\mathcal{F}}_{\varepsilon,b}\right], \tag{3.3}$$

has a well-defined iterative solution in the regime (i) in which $0 < \bar{\alpha}_s(t) < \varepsilon/b$. In fact, the $\boldsymbol{k}$-integrations are convergent in the IR because of $\bar{\alpha}_s(t) \sim e^{\varepsilon t}$ for $t \to -\infty$, and everywhere else because $\bar{\alpha}_s(t)$ is bounded. In this sense, $b, \varepsilon > 0$ in the regime (i) act as regulators of both the IR and UV regions, and of the Landau pole, so that eq. (3.2) is meaningful without any cutoffs — a somewhat unique case in BFKL theory. In the $b \to 0$ limit eq. (3.2) reduces to eq. (2.2), but becomes less convergent in the UV region: as noticed in appendix A.2, only a finite number of iterations $n < 1/\varepsilon$ is actually possible for $b \neq 0$, for the UV integrations to be convergent.

Therefore we shall solve eq. (3.2) in the regime (i), where it is nicely convergent, and we shall study the factorisation of $1/\varepsilon$ singularities by letting $b, \varepsilon \to 0$ with $\varepsilon/b > \bar{\alpha}_s$ kept fixed. Eventually, we are instead interested in the physical limit of $\varepsilon \to 0$ with $\bar{\alpha}_s$ and b kept fixed, but we shall perform it only at the end, after factorisation of the $1/\varepsilon$ poles.

Let us introduce the γ-representation (2.7) into eq. (3.3) after multiplying it by the t-dependent denominator. After simple algebra we get the equation

$$\begin{aligned} f_{\varepsilon,b}(\gamma+\varepsilon) &= \frac{1}{\omega}\left(\frac{1}{\bar{\alpha}_s}-\frac{b}{\varepsilon}\right)^{-1}\left[\frac{e^{\gamma T}}{\gamma}+\left(\chi_\varepsilon(\gamma)-\frac{b\omega}{\varepsilon}\right)f_{\varepsilon,b}(\gamma)\right] \\ &\equiv f^{(0)}_{\varepsilon,b}(\gamma)+\chi_{\varepsilon,b}(\gamma)f_{\varepsilon,b}(\gamma)\,, \end{aligned} \tag{3.4}$$

where we have again introduced the cutoff $t > -T$ in the initial condition (cf. eq. (2.10)), in order to better control the $\varepsilon \to 0$ limit. Eq. (3.4) has the form of eq. (2.8) and admits a similar iterative solution, which is obtained from eq. (2.24) by the replacements

$$\bar{\alpha}_s \to \left(\frac{1}{\bar{\alpha}_s}-\frac{b}{\varepsilon}\right)^{-1}, \tag{3.5a}$$

$$f^{(0)}(\gamma) \to f^{(0)}_{\varepsilon,b}(\gamma) \equiv \frac{1}{\omega}\left(\frac{1}{\bar{\alpha}_s}-\frac{b}{\varepsilon}\right)^{-1}\frac{e^{\gamma T}}{\gamma}, \tag{3.5b}$$

$$\chi_\varepsilon(\gamma) \to \chi_{\varepsilon,b}(\gamma) \equiv \chi_\varepsilon(\gamma) - \frac{b\omega}{\varepsilon}\,. \tag{3.5c}$$

In the limit of $b, \varepsilon \to 0$ with b/ε and the cutoff T kept fixed we get identically from eqs. (2.24) and (3.5) the frozen coupling solution (2.25) and (2.26), as expected.

We are more interested, however, in the continuum limit of the solution of eq. (3.4) in the regime (i) and for $T \to +\infty$. By formal manipulations similar to the $b = 0$ case we obtain

$$\widetilde{\mathcal{F}}_{\varepsilon,b}(\boldsymbol{k}) \overset{T\to+\infty}{\longrightarrow} \int \frac{d\gamma}{\sqrt{2\pi\varepsilon}}\,\frac{\exp\left\{\gamma t + \frac{1}{\varepsilon}\int_0^\gamma L_{0,b}(\gamma')d\gamma' + \int_0^\gamma \frac{\chi_1(\gamma')}{\chi_0(\gamma')-\frac{b\omega}{\varepsilon}}d\gamma' + \mathcal{O}(\varepsilon)\right\}}{\sqrt{\chi_0(\gamma)-\frac{b\omega}{\varepsilon}}}, \tag{3.6}$$

which is again a solution of the homogeneous equation, where

$$L_{\varepsilon,b}(\gamma) \equiv \log\left(\frac{\frac{1}{\omega}\chi_\varepsilon(\gamma) - \frac{b}{\varepsilon}}{\frac{1}{\bar{\alpha}_\mathrm{s}} - \frac{b}{\varepsilon}}\right), \quad L_{0,b}(\gamma) \equiv \log\left(\frac{\frac{1}{\omega}\chi_0(\gamma) - \frac{b}{\varepsilon}}{\frac{1}{\bar{\alpha}_\mathrm{s}} - \frac{b}{\varepsilon}}\right) \tag{3.7}$$

and the exponent has been expanded in ε (at $b\bar{\alpha}_\mathrm{s}/\varepsilon$ fixed) up to the finite terms. Once again, the factorisation of the $1/\varepsilon$ poles in (3.6) is investigated, for $\varepsilon \to 0$, by the saddle point condition

$$\varepsilon t + L_{0,b}(\bar{\gamma}) = 0 \qquad \Longleftrightarrow \qquad 1 = \frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\chi_0(\bar{\gamma}), \tag{3.8}$$

where, due to the b-dependence of eq. (3.7), $\bar{\alpha}_\mathrm{s}(t)$ has the expected form (3.1). It follows that

$$\widetilde{\mathcal{F}}_{\varepsilon,b}(\boldsymbol{k}) = \frac{1}{\sqrt{-\chi_0'(\bar{\gamma}_t)}} \exp\left\{\bar{\gamma}_t\, t + \frac{1}{\varepsilon}\int_0^{\bar{\gamma}_t} L_{0,b}(\gamma')\mathrm{d}\gamma' + \int_0^{\bar{\gamma}_t} \frac{\chi_1(\gamma')}{\chi_0(\gamma') - \frac{b\omega}{\varepsilon}}\mathrm{d}\gamma'\right\} \times \left[1 + \mathcal{O}(\varepsilon)\right] \tag{3.9}$$

and that

$$\begin{aligned} g_{\varepsilon,b}(t) &= \frac{1}{\bar{\gamma}_t\sqrt{-\chi_0'(\bar{\gamma}_t)}} \exp\left\{\bar{\gamma}_t\, t + \frac{1}{\varepsilon}\int_0^{\bar{\gamma}_t} L_{0,b}(\gamma')\mathrm{d}\gamma' + \int_0^{\bar{\gamma}_t} \frac{\chi_1(\gamma')}{\chi_0(\gamma') - \frac{b\omega}{\varepsilon}}\mathrm{d}\gamma'\right\} \times \left[1 + \mathcal{O}(\varepsilon)\right] \\ &= \mathcal{N}\Big(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\Big) \exp\left\{\int_{-\infty}^t \mathrm{d}\tau\, \left[\gamma_0\Big(\frac{\bar{\alpha}_\mathrm{s}(\tau)}{\omega}\Big) + \varepsilon\gamma_1\Big(\frac{\bar{\alpha}_\mathrm{s}(\tau)}{\omega}\Big)\right]\right\} \times \left[1 + \mathcal{O}(\varepsilon)\right]. \end{aligned} \tag{3.10}$$

The final expression in eq. (3.10) is formally identical to eq. (2.39) and is expected to have analogous factorisation properties. However, due to the different form of $\bar{\alpha}_\mathrm{s}(t)$, the jacobians induced by eq. (3.8) are different:

$$\mathrm{d}t = \frac{\mathrm{d}\bar{\alpha}_\mathrm{s}}{\bar{\alpha}_\mathrm{s}(\varepsilon - b\bar{\alpha}_\mathrm{s})} = -\frac{\chi_0'(\bar{\gamma}_t)}{\varepsilon\left[\chi_0(\bar{\gamma}_t) - \frac{b\omega}{\varepsilon}\right]}\mathrm{d}\bar{\gamma}_t\,, \tag{3.11}$$

and this explains the different ε-dependence of the exponents at fixed value of $\bar{\alpha}_\mathrm{s}(t)$:

$$\int_{-\infty}^t \mathrm{d}\tau\; \gamma_0\Big(\frac{\bar{\alpha}_\mathrm{s}(\tau)}{\omega}\Big) = \int_0^{\bar{\alpha}_\mathrm{s}(t)} \frac{\mathrm{d}\alpha}{\alpha(\varepsilon - b\alpha)}\; \gamma_0\Big(\frac{\alpha}{\omega}\Big) = \bar{\gamma}_t\, t + \frac{1}{\varepsilon}\int_0^{\bar{\gamma}_t} \mathrm{d}\gamma'\; L_{0,b}(\gamma') \tag{3.12}$$

and

$$\varepsilon\int_{-\infty}^t \mathrm{d}\tau\; \gamma_1\Big(\frac{\bar{\alpha}_\mathrm{s}(\tau)}{\omega}\Big) = \varepsilon\int_0^{\bar{\alpha}_\mathrm{s}(t)} \frac{\mathrm{d}\alpha}{\alpha(\varepsilon - b\alpha)}\; \gamma_1\Big(\frac{\alpha}{\omega}\Big). \tag{3.13}$$

The factorisation of the t-dependence in eq. (3.10) from the $1/\varepsilon$ singularities is now performed as in section 2 by subdividing the τ-integration into the $]-\infty, t_f]$ and $[t_f, t]$ intervals, where $t_f \equiv \log(\mu_f^2/\mu^2)$ defines the factorisation scale. The two intervals are treated differently: in the finite UV interval $[t_f, t]$ we can freely go to the $\varepsilon = 0$ limit at fixed value of $\bar{\alpha}_\mathrm{s}(t_f)$, thus recovering the $\bar{\alpha}_\mathrm{s}(t) \to \bar{\alpha}_\mathrm{s}(t_f)/[1 + b\bar{\alpha}_\mathrm{s}(t_f)(t - t_f)]$ limit and the normal t-dependence of UV free QCD in 4 dimensions. In the remaining infinite IR interval we have to factorise the $1/\varepsilon$ poles by expanding in the $b\bar{\alpha}_\mathrm{s}/\varepsilon$ parameter, as normally done in fixed order perturbation theory.

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By looking at eqs. (3.12) and (3.13) (with t replaced by t_f) we realise the following: eq. (3.12) exponentiates $1/\varepsilon$ poles and higher order ones coming from the $b\bar{\alpha}_s/\varepsilon$-expansion, and has to be factorised in full in the $\overline{\text{MS}}$-scheme; it evolves in t_f according to the LLx anomalous dimension. On the other hand, the term (3.13) — due to the $\mathcal{O}(\varepsilon)$ correction to the BFKL anomalous dimension — reduces, in the $b \to 0$ limit, to the $\mathfrak{R}\big(\bar{\alpha}_s(t_f)/\omega\big)$ factor found in section 2, which yields $R/\mathcal{N}$, i.e., the normalisation mismatch of the frozen coupling evolution with respect to the $\overline{\text{MS}}$ density. Furthermore, in the $b\bar{\alpha}_s/\varepsilon$-expansion, eq. (3.13) exponentiates $1/\varepsilon$ and higher order poles which should be factored out in the $\overline{\text{MS}}$ density, and therefore contribute to the $\overline{\text{MS}}$ anomalous dimension.

In order to perform collectively such separation, we rewrite the r.h.s. of eq. (3.13) (with $t = t_f$) in the form

$$\int_0^{\bar{\alpha}_s(t_f)} \frac{d\alpha}{\alpha}\,\gamma_1\Big(\frac{\alpha}{\omega}\Big) + \int_0^{\bar{\alpha}_s(t_f)} \frac{d\alpha}{\alpha(\varepsilon - b\alpha)}\, b\alpha\,\gamma_1\Big(\frac{\alpha}{\omega}\Big)\,, \tag{3.14}$$

where the first term provides $\log\mathfrak{R}\big(\bar{\alpha}_s(t_f)/\omega\big)$, and the second one is the anomalous dimension contribution to be factored out in the $\overline{\text{MS}}$ density. Finally, by replacing eq. (3.14) into eq. (3.10), we find

$$g_\varepsilon(t) = \mathcal{N}\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)\exp\left\{\int_{t_f}^{t} d\tau\,\gamma_0\Big(\frac{\bar{\alpha}_s(\tau)}{\omega}\Big)\right\}\mathfrak{R}\Big(\frac{\bar{\alpha}_s(t_f)}{\omega}\Big) g^{(\overline{\text{MS}})}(t_f)\,, \tag{3.15}$$

where

$$g^{(\overline{\text{MS}})}(t_f) = \exp\left\{\int_0^{\bar{\alpha}_s(t_f)} \frac{d\alpha}{\alpha(\varepsilon - b\alpha)}\,\Big[\gamma_0\Big(\frac{\alpha}{\omega}\Big) + b\alpha\gamma_1\Big(\frac{\alpha}{\omega}\Big)\Big]\right\} \tag{3.16}$$

incorporates all the $1/\varepsilon$ poles, in a formal $b\bar{\alpha}_s/\varepsilon$-expansion.

Eqs. (3.15) and (3.16) are the main results of this section. They confirm the relation between our generalised Q_0-scheme (with dimensional regularisation) and the $\overline{\text{MS}}$-scheme, namely

$$g^{(Q_0)}(t) \equiv g_\varepsilon(t) = R\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) g^{(\overline{\text{MS}})}(t)\,, \tag{3.17}$$

and they prove the following resummation formulas

$$\gamma^{(Q_0)}\big(\bar{\alpha}_s(t);\omega\big) \equiv \frac{\frac{d}{dt}g^{(Q_0)}(t)}{g^{(Q_0)}(t)} = \gamma_0\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) - b\bar{\alpha}_s(t)\frac{\partial\log\mathcal{N}}{\partial\log\bar{\alpha}_s} \tag{3.18}$$

$$\begin{aligned}\gamma^{(\overline{\text{MS}})}\big(\bar{\alpha}_s(t);\omega\big) \equiv \frac{\frac{d}{dt}g^{(\overline{\text{MS}})}(t)}{g^{(\overline{\text{MS}})}(t)} &= \gamma_0\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) + b\bar{\alpha}_s(t)\frac{\partial\log\mathfrak{R}}{\partial\log\bar{\alpha}_s} \\ &= \gamma_0\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) + b\bar{\alpha}_s(t)\gamma_1\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)\,,\end{aligned} \tag{3.19}$$

where $\mathcal{N}$, $\mathfrak{R}$, γ_0 and γ_1 are defined in eqs. (2.40), (2.45), (2.27) and (2.41) respectively.

Let us remark that the results (3.18)–(3.19) are not surprising, in view of the well studied relationship of the $\mathcal{N}$ and R factors already performed in the literature in a somewhat different context [10]. However they offer some important insights into the explicit RG factorisation obtained here: eq. (3.18) shows that the evolution of the gluon density in a $\boldsymbol{k}$-factorisation scheme is independent of whether the BFKL equation is regularised by a cutoff (Q_0-scheme in strict sense) or by a positive ε (as done at present). This is a consequence of the RG factorisation just proved. Eq. (3.19) shows that the $\overline{\mathrm{MS}}$ evolution is the one expected because of the R factor in eq. (3.17). However, this result is obtained by the explicit, b-dependent factorisation of the $1/\varepsilon$ poles connected with γ_1 in eq. (3.16). Furthermore the t_f-dependence in eq. (3.15) cancels out because of the γ_1 evolution in $g^{(\overline{\mathrm{MS}})}$ cancelling the t_f-dependence of $\mathfrak{R}$. All together, such terms build up the $\mathcal{O}(\varepsilon)$ correction to the BFKL anomalous dimension (as done in eq. (3.14)), which precisely vanishes in the $\varepsilon \to 0$ limit.

Our next goal is to generalise the above factorisation procedure to subleading terms, in particular to the coefficient terms at NLx level and to the corresponding NNLx anomalous dimension terms.

4. Factorisation at subleading level: b-dependent corrections

We have just realised that the normalisation factor $\mathfrak{R}$ arises from the $\mathcal{O}(\varepsilon)$ correction to the BFKL anomalous dimension (3.13) after the (minimal) subtraction of a b-dependent contribution to the $\overline{\mathrm{MS}}$ anomalous dimension (second term of (3.14)). This suggests that, in order to compute corrections of relative order $b\bar{\alpha}_\mathrm{s}$ to $\mathfrak{R}$ and the corresponding ones to $\gamma^{(\overline{\mathrm{MS}})}$ we have to calculate $\mathcal{O}\left(\varepsilon^2\right)$ corrections to the BFKL anomalous dimension and, more generally, to the exponent of the solution (3.6) of the homogeneous equation. We thus restart our analysis from the "action" (2.17) and its b/ε-dependent counterpart, and we rewrite the solution (3.6) of the homogeneous equation in the form

$$\widetilde{\mathcal{F}}_{\varepsilon,b}(\boldsymbol{k}) = \int \frac{\mathrm{d}\gamma}{\sqrt{2\pi\varepsilon}} \frac{1}{\sqrt{\chi_\varepsilon(\gamma) - \frac{b\omega}{\varepsilon}}} \exp\left\{\gamma t + \frac{1}{\varepsilon}\int_0^\gamma L_{\varepsilon,b}(\gamma')\mathrm{d}\gamma' + \frac{\varepsilon}{12} L'_{\varepsilon,b}(\gamma) + \mathcal{O}\left(\varepsilon^2\right)\right\}, \tag{4.1}$$

where we have kept the ε-dependence of L and χ, and the $\mathcal{O}(\varepsilon)$ correction to the "action" (which is of relative order $\mathcal{O}\left(\varepsilon^2\right)$).

We then look at the factorisation properties of eq. (4.1) by expanding around the ε-dependent saddle point $\gamma = \bar{\gamma}_\varepsilon$:

$$\varepsilon t + L_{\varepsilon,b}(\bar{\gamma}_\varepsilon) = 0 \qquad \Longleftrightarrow \qquad 1 = \frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\chi_\varepsilon(\bar{\gamma}_\varepsilon)\,, \tag{4.2}$$

which, therefore, defines the ε-dependent BFKL anomalous dimension

$$\gamma_\varepsilon\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) = \gamma_0\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) + \varepsilon\gamma_1\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) + \varepsilon^2\gamma_2\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) + \cdots\,, \tag{4.3}$$

where γ_1 was given before in eq. (2.41), and we have[5]

$$\gamma_2\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) = -\frac{\chi_2(\gamma)}{\chi'_0(\gamma)} + \frac{\chi_1(\gamma)\chi'_1(\gamma)}{{\chi'_0}^2(\gamma)} - \frac{1}{2}\frac{\chi_1^2(\gamma)\chi''_0(\gamma)}{{\chi'_0}^3(\gamma)}\bigg|_{\gamma=\gamma_0\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)}\,. \tag{4.4}$$

[5]We recall that the subscripts in the anomalous dimension γ and in the eigenvalue function χ refer to the expansion in the ε variable (cf. eqs. (2.19)–(2.22)). The coefficients of the N^{n}Lx expansions will instead be labelled by superscripts (cf. section 5).

Furthermore, we have to compute the γ-integral by expanding the action and the factor $\left[\chi_\varepsilon(\gamma) - \frac{b\omega}{\varepsilon}\right]^{-1/2}$ around the saddle point. Since the γ-fluctuations are governed by the width $\sigma_\gamma = \left[-\varepsilon/L'_{\varepsilon,b}(\bar{\gamma}_\varepsilon)\right]^{1/2}$ which is of $\mathcal{O}\left(\sqrt{\varepsilon}\right)$ while the "action" is $\mathcal{O}\left(1/\varepsilon\right)$, we need to expand, it turns out, up to 6-th order in $\gamma - \bar{\gamma}$ in order to reach all $\mathcal{O}\left(\varepsilon\right)$ terms. This calculation is performed in appendix C and, when replaced into eq. (4.1), provides the following result for the gluon density:

$$g_\varepsilon(t) = \frac{1}{\bar{\gamma}_\varepsilon\sqrt{-\chi'_\varepsilon(\bar{\gamma}_\varepsilon)}} \exp\left\{\int_{-\infty}^{t} \mathrm{d}\tau\; \bar{\gamma}_\varepsilon\Big(\frac{\bar{\alpha}_\mathrm{s}(\tau)}{\omega}\Big)\right\} \times \left[1 + \varepsilon S_1(\bar{\gamma}_\varepsilon, b)\right]. \tag{4.5}$$

Here we have used the identity

$$\bar{\gamma}_\varepsilon\, t + \frac{1}{\varepsilon}\int_0^{\bar{\gamma}_\varepsilon} \mathrm{d}\gamma'\; L_{\varepsilon,b}(\gamma') = \int_{-\infty}^{t} \mathrm{d}\tau\; \bar{\gamma}_\varepsilon\Big(\frac{\bar{\alpha}_\mathrm{s}(\tau)}{\omega}\Big), \tag{4.6}$$

the jacobians

$$\mathrm{d}t = \frac{\mathrm{d}\bar{\alpha}_\mathrm{s}}{\bar{\alpha}_\mathrm{s}(\varepsilon - b\bar{\alpha}_\mathrm{s})} = -\frac{\chi'_\varepsilon(\bar{\gamma}_\varepsilon)}{\varepsilon\left[\chi_\varepsilon(\bar{\gamma}_\varepsilon) - \frac{b\omega}{\varepsilon}\right]}\mathrm{d}\bar{\gamma}_\varepsilon\,, \tag{4.7}$$

and the $\mathcal{O}\left(\varepsilon\right)$ correction to the action

$$S_1(\gamma, b) = \frac{1}{12}L'_{\varepsilon,b} + \left[\frac{1}{8}(-L'_{\varepsilon,b}) + \frac{5}{24}\frac{{L''_{\varepsilon,b}}^2}{(-L'_{\varepsilon,b})^3} + \frac{1}{8}\frac{L'''_{\varepsilon,b}}{{L'_{\varepsilon,b}}^2}\right], \tag{4.8}$$

where the terms in square brackets are precisely the fluctuations calculated in appendix C.1.

Let us now look at the factorisation properties of our result in eq. (4.5) in the limit of $\varepsilon, b\bar{\alpha}_\mathrm{s}/\varepsilon \to 0$. The anomalous dimension exponential is factorised as usual, and its infrared part at the factorisation scale t_f is given by

$$\int_0^{\bar{\alpha}_\mathrm{s}(t_f)} \frac{\mathrm{d}\alpha}{\alpha(\varepsilon - b\alpha)}\left[\gamma_0\Big(\frac{\alpha}{\omega}\Big) + \varepsilon\gamma_1\Big(\frac{\alpha}{\omega}\Big) + \varepsilon^2\gamma_2\Big(\frac{\alpha}{\omega}\Big) + \cdots\right]. \tag{4.9}$$

We have now the additional $\mathcal{O}\left(\varepsilon^2\right)$ term γ_2, which however still builds $1/\varepsilon$ singularities because of the $b\bar{\alpha}_\mathrm{s}/\varepsilon$-expansion of the β-function in the denominator. The decomposition into coefficient and $\overline{\mathrm{MS}}$-anomalous dimension parts is simply done by writing $\varepsilon^2 = (b\bar{\alpha}_\mathrm{s})^2 + (\varepsilon - b\bar{\alpha}_\mathrm{s})(b\bar{\alpha}_\mathrm{s} + \varepsilon)$, so that

$$\int_0^{\bar{\alpha}_\mathrm{s}(t_f)} \frac{\mathrm{d}\alpha}{\alpha(\varepsilon - b\alpha)}\varepsilon^2\gamma_2\Big(\frac{\alpha}{\omega}\Big) = \int_0^{\bar{\alpha}_\mathrm{s}(t_f)} \frac{\mathrm{d}\alpha}{\alpha(\varepsilon - b\alpha)}(b\alpha)^2\gamma_2\Big(\frac{\alpha}{\omega}\Big) + \int_0^{\bar{\alpha}_\mathrm{s}(t_f)} \frac{\mathrm{d}\alpha}{\alpha}b\alpha\gamma_2\Big(\frac{\alpha}{\omega}\Big) + \mathcal{O}\left(\varepsilon\right), \tag{4.10}$$

where the first term in the r.h.s. is the minimal subtraction part of the γ_2 anomalous dimension, the second one is the $\mathcal{O}\left(b\bar{\alpha}_\mathrm{s}\right)$ correction to $\log\mathfrak{R}$.

We have also to look for $1/\varepsilon$ singularities possibly arising because of the $b\bar{\alpha}_\mathrm{s}/\varepsilon$-expansion in the contribution (4.8). Here the situation is not so clear *a priori*, given the fact that

$$-L'_{\varepsilon,b}(\bar{\gamma}_\varepsilon) = \frac{-\chi'_\varepsilon(\bar{\gamma}_\varepsilon)}{\chi_\varepsilon(\bar{\gamma}_\varepsilon) - \frac{b\omega}{\varepsilon}} = -\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\frac{\chi'_\varepsilon\big(\bar{\gamma}_\varepsilon(t)\big)}{1 - \frac{b\bar{\alpha}_\mathrm{s}(t)}{\varepsilon}} \tag{4.11}$$

has a non-trivial $b\bar{\alpha}_s/\varepsilon$ expansion. However, after some algebra, we find a remarkable cancellation leading to the result (see appendix C.2)

$$S_1(\bar{\gamma}_\varepsilon(t), b) = \frac{1}{24}\left\{\frac{\chi''_\varepsilon}{\chi'_\varepsilon} + \frac{\omega}{\chi'_\varepsilon}\left(\frac{b}{\varepsilon} - \frac{1}{\bar{\alpha}_s(t)}\right)\left[2\left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)^2 - 3\left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)'\right]\right\}, \tag{4.12}$$

which is only linear in b/ε. It follows that S_1 has no minimal subtraction terms, and only contributes to the renormalisation of the $\mathcal{N}$-factor, as follows:

$$\varepsilon S_1(\bar{\gamma}_\varepsilon(t), b) = \frac{b\omega}{24\chi'_0}\left[2\left(\frac{\chi''_0}{\chi'_0}\right)^2 - 3\left(\frac{\chi''_0}{\chi'_0}\right)'\right] + \mathcal{O}(\varepsilon)\,. \tag{4.13}$$

We can thus summarise our results on the NLx corrections to the coefficient factors of eq. (3.15) (and on the NNLx ones to the $\overline{\text{MS}}$ anomalous dimension). The $\mathfrak{R}$ factor takes the form

$$\mathfrak{R}(\bar{\alpha}_s(t_f);\omega) = \exp\left\{\int_0^{\bar{\alpha}_s(t_f)} \frac{\mathrm{d}\alpha}{\alpha}\,\left[\gamma_1\left(\frac{\alpha}{\omega}\right) + b\alpha\,\gamma_2\left(\frac{\alpha}{\omega}\right) + \cdots\right]\right\}, \tag{4.14}$$

and the $\overline{\text{MS}}$ anomalous dimension becomes

$$\begin{aligned}\gamma^{(\overline{\text{MS}})}(\bar{\alpha}_s;\omega) &= \gamma_0\left(\frac{\bar{\alpha}_s}{\omega}\right) + b\bar{\alpha}_s\,\gamma_1\left(\frac{\bar{\alpha}_s}{\omega}\right) + b^2\bar{\alpha}_s^2(t)\,\gamma_2\left(\frac{\bar{\alpha}_s}{\omega}\right) + \cdots \\ &= \gamma_0\left(\frac{\bar{\alpha}_s}{\omega}\right) + b\bar{\alpha}_s\frac{\partial\log\mathfrak{R}(\bar{\alpha}_s;\omega)}{\partial\log\bar{\alpha}_s}\,.\end{aligned} \tag{4.15}$$

Due to the form of (4.15) and (4.14), the t_f-dependence of the factorisation formula (3.15) cancels out as it should, and as is expected on the basis of their common origin, eq. (4.9). We also expect this mechanism to hold true to all orders in $b\bar{\alpha}_s$, so that the $\overline{\text{MS}}$ anomalous dimension and coefficient can be inferred from the corresponding ε-expansion of the BFKL anomalous dimension (4.3).

On the other hand, the fluctuation factor $\mathcal{N}$ takes finite NLx corrections from eq. (4.13), and becomes

$$\mathcal{N}(\bar{\alpha}_s;\omega) = \frac{1}{\gamma_0\sqrt{-\chi'_0}}\left\{1 + \frac{b\omega}{24\chi'_0}\left[2\left(\frac{\chi''_0}{\chi'_0}\right)^2 - 3\left(\frac{\chi''_0}{\chi'_0}\right)'\right] + \cdots\right\}. \tag{4.16}$$

Such corrections are identical to those found from the normal γ-representation in 4 dimensions. In this case also we expect that higher orders in the ε-expansion (2.17) and the corresponding fluctuations will combine so as to provide further finite subleading corrections to eq. (4.16). We have no formal proof of this expectation, but we notice that higher order fluctuations still involve the scale t and the coupling $\bar{\alpha}_s(t)$ and are thus independent of the IR behaviour at $\tau \to -\infty$. It is then natural to believe that they can be computed directly in the $\varepsilon = 0$ limit from the normal γ-representation in 4 dimensions, as explicitly proved above for the next-to-leading terms.

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5. On the treatment of the full NLx corrections to the $\mathfrak{R}$ factor

Having calculated the running coupling corrections to $\mathfrak{R}$, the problem arises of including *all* NLx contributions, as embodied in the next-to-leading BFKL kernel. The structure and the explicit expression of such a kernel with dimensional regularisation were given in several papers for the gluon [7] and the quark [8, 9] parts, and summarised in Refs. [10 – 12]. It has the form

$$\mathcal{K}_\varepsilon^{(\mathrm{NL})} = \bar{\alpha}_\mathrm{s}\left[\frac{b}{\varepsilon}\left(1-\mathrm{e}^{\varepsilon t}\right)\mathcal{K}_\varepsilon\right] + \mathcal{K}_\varepsilon^{(1)}\,, \tag{5.1}$$

where we have singled out the running coupling part (proportional to the leading kernel $\mathcal{K}_\varepsilon$) which arises from the $\mathcal{O}\left(\bar{\alpha}_\mathrm{s}^2\right)$ expansion of the running coupling equation (3.2). The remaining kernel $\mathcal{K}_\varepsilon^{(1)}$, which scales as $\mathrm{e}^{2\varepsilon t}$, is the properly called NLx kernel, whose eigenvalues have been worked out in the literature in the $\varepsilon \to 0$ limit, and are here required up to $\mathcal{O}(\varepsilon)$ for the complete calculation of NLx corrections to $\mathfrak{R}$.

We shall thus generalise eq. (3.2) to include the NLx kernel in the form

$$\begin{aligned}\mathcal{F}_\varepsilon(\boldsymbol{k}) &= \delta^{(2+2\varepsilon)}(\boldsymbol{k}) + \frac{1}{\omega}\,\frac{1}{1+b\bar{\alpha}_\mathrm{s}\frac{\mathrm{e}^{\varepsilon t}-1}{\varepsilon}}\int\frac{\mathrm{d}^{2+2\varepsilon}\boldsymbol{k}'}{(2\pi)^{2+2\varepsilon}}\left[\mathcal{K}_\varepsilon(\boldsymbol{k},\boldsymbol{k}') + \mathcal{K}_\varepsilon^{(1)}(\boldsymbol{k},\boldsymbol{k}')\right]\mathcal{F}_\varepsilon(\boldsymbol{k}') \\ &= \delta^{(2+2\varepsilon)}(\boldsymbol{k}) + \frac{\mathrm{e}^{\varepsilon\psi(1)}}{(\pi\boldsymbol{k}^2)^{1+\varepsilon}}\widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k})\,.\end{aligned} \tag{5.2}$$

where NNLx terms and further subleading ones have been freely added so as to reproduce the resummed $\bar{\alpha}_\mathrm{s}(t)$ evolution.[6] It is now straightforward to go to γ-space and to obtain a modified form of eq. (3.4). By introducing the NLx "characteristic function" $\chi_\varepsilon^{(1)}$ of $\mathcal{K}_\varepsilon^{(1)}$ in $4+2\varepsilon$ dimensions

$$\int\frac{\mathrm{d}^{2+2\varepsilon}\boldsymbol{k}'}{(2\pi)^{2+2\varepsilon}}\,\mathcal{K}_\varepsilon^{(1)}(\boldsymbol{k},\boldsymbol{k}')(\boldsymbol{k}'^2)^{\gamma-1-2\varepsilon} \equiv \bar{\alpha}_\mathrm{s}^2\chi_\varepsilon^{(1)}(\gamma)\frac{(\boldsymbol{k}^2)^{\gamma-1}}{\mu^{4\varepsilon}}\,, \tag{5.3}$$

the corresponding homogeneous equation reads

$$f_\varepsilon(\gamma+\varepsilon) - \frac{b\bar{\alpha}_\mathrm{s}}{\varepsilon}\left[f_\varepsilon(\gamma+\varepsilon) - f_\varepsilon(\gamma)\right] = \bar{\alpha}_\mathrm{s}\frac{\chi_\varepsilon(\gamma)}{\omega}f_\varepsilon(\gamma) + \frac{\bar{\alpha}_\mathrm{s}^2}{\omega}\chi_\varepsilon^{(1)}(\gamma)f_\varepsilon(\gamma-\varepsilon)\,, \tag{5.4}$$

and contains, therefore, two finite difference steps, due to the different scaling properties of the leading vs. next-to-leading kernels. However, to NLx accuracy, we can replace the leading order equation in the last term, and we obtain

$$f_\varepsilon(\gamma+\varepsilon) = \left[\frac{\frac{\chi_\varepsilon(\gamma)}{\omega} - \frac{b}{\varepsilon} + \frac{\chi_\varepsilon^{(1)}(\gamma)}{\chi_\varepsilon(\gamma-\varepsilon)}}{\frac{1}{\bar{\alpha}_\mathrm{s}} - \frac{b}{\varepsilon}}\right]f_\varepsilon(\gamma) \equiv \exp[L_{\varepsilon,b}^{\mathrm{eff}}(\gamma)]f_\varepsilon(\gamma)\,, \tag{5.5}$$

where the next-to-leading term is now suppressed by a factor of ω with respect to the leading one, much in the same spirit as the ω-expansion [18].

[6] Here we keep terms which are of relative order $\bar{\alpha}_\mathrm{s}\varepsilon^n$ $(n \geq 0)$ with respect to the leading ones, and we drop terms of relative order $b\bar{\alpha}_\mathrm{s}^2$ or higher, at fixed values of ε.

We can thus solve eq. (5.5) by the same method used before to get eq. (4.1). Correspondingly, the saddle point at $\gamma = \bar{\gamma}_\varepsilon\big(\bar{\alpha}_\mathrm{s}(t);\omega\big)$ such that

$$\varepsilon t + L^{\mathrm{eff}}_{\varepsilon,b}(\bar{\gamma}_\varepsilon) = 0 \qquad \Longleftrightarrow \qquad \bar{\alpha}_\mathrm{s}(t)\left[\chi_\varepsilon(\bar{\gamma}_\varepsilon) + \omega\frac{\chi^{(1)}_\varepsilon(\bar{\gamma}_\varepsilon)}{\chi_\varepsilon(\bar{\gamma}_\varepsilon - \varepsilon)}\right] = \omega \tag{5.6}$$

admits the solution

$$\bar{\gamma}_\varepsilon = \gamma^{(0)}_\varepsilon\left(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\right) + \bar{\alpha}_\mathrm{s}(t)\gamma^{(1)}_\varepsilon\left(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\right) + \mathcal{O}\left(\bar{\alpha}_\mathrm{s}^2\right)\,, \tag{5.7}$$

where $\gamma^{(0)}_\varepsilon$ defines the LLx BFKL anomalous dimension of eq. (4.3) and $\gamma^{(1)}_\varepsilon$ is its NLx correction, obtained by truncating the expansion of the saddle point position to relative order $\mathcal{O}\left(\bar{\alpha}_\mathrm{s}(t)\right)$:

$$\gamma^{(1)}_\varepsilon\left(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\right) = -\frac{\chi^{(1)}_\varepsilon(\gamma^{(0)}_\varepsilon)}{\chi'_\varepsilon(\gamma^{(0)}_\varepsilon)}\frac{\chi_\varepsilon(\gamma^{(0)}_\varepsilon)}{\chi_\varepsilon(\gamma^{(0)}_\varepsilon - \varepsilon)} = \gamma^{(1)}_0\left(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\right) + \varepsilon\gamma^{(1)}_1\left(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\right) + \mathcal{O}\left(\varepsilon^2\right)\,. \tag{5.8}$$

This expression reduces to the customary one [11, 12] in the $\varepsilon = 0$ limit, but has $\mathcal{O}(\varepsilon)$ corrections coming from the corresponding ones of $\chi^{(1)}_\varepsilon$ — yet to be extracted from the various papers in the literature [7–9].

Finally, we expand the ε-dependence of the BFKL anomalous dimension, including NLx terms, as follows:

$$\bar{\gamma}_\varepsilon(\bar{\alpha}_\mathrm{s};\omega) = \gamma^{(0)}_0\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) + \bar{\alpha}_\mathrm{s}\gamma^{(1)}_0\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) + \varepsilon\gamma^{(0)}_1\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) + \varepsilon\left[\bar{\alpha}_\mathrm{s}\gamma^{(1)}_1\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right) + \varepsilon\gamma^{(0)}_2\left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)\right] \tag{5.9}$$

and we replace it into the analogue of eq. (4.5). The corresponding anomalous dimension exponential factorises in the form

$$\exp\left\{\int_0^{\bar{\alpha}_\mathrm{s}(t)} \frac{\mathrm{d}\alpha}{\alpha(\varepsilon - b\alpha)}\;\bar{\gamma}_\varepsilon(\alpha;\omega)\right\} = \mathfrak{R}\big(\bar{\alpha}_\mathrm{s}(t);\omega\big)g^{(\overline{\mathrm{MS}})}(t)\,, \tag{5.10}$$

where

$$\mathfrak{R}\big(\bar{\alpha}_\mathrm{s}(t);\omega\big) = \exp\left\{\int_0^{\bar{\alpha}_\mathrm{s}(t)} \frac{\mathrm{d}\alpha}{\alpha}\;\left[\gamma^{(0)}_1\left(\frac{\alpha}{\omega}\right) + \alpha\gamma^{(1)}_1\left(\frac{\alpha}{\omega}\right) + b\alpha\gamma^{(0)}_2\left(\frac{\alpha}{\omega}\right)\right]\right\} \tag{5.11}$$

and

$$g^{(\overline{\mathrm{MS}})}(t) = \exp\left\{\int_0^{\bar{\alpha}_\mathrm{s}(t)} \frac{\mathrm{d}\alpha}{\alpha(\varepsilon - b\alpha)}\;\bar{\gamma}_{\varepsilon=b\alpha}(\alpha;\omega)\right\}\,. \tag{5.12}$$

It follows that the $\overline{\mathrm{MS}}$ gluon anomalous dimension is

$$\begin{aligned}\gamma^{(\overline{\mathrm{MS}})}(\bar{\alpha}_\mathrm{s};\omega) = \bar{\gamma}_{\varepsilon=b\bar{\alpha}_\mathrm{s}}(\bar{\alpha}_\mathrm{s};\omega) &= \gamma^{(0)}_0 + \bar{\alpha}_\mathrm{s}\gamma^{(1)}_0 + b\bar{\alpha}_\mathrm{s}\gamma^{(0)}_1 + b\bar{\alpha}_\mathrm{s}^2\gamma^{(1)}_1 + (b\bar{\alpha}_\mathrm{s})^2\gamma^{(0)}_2 \\ &= \gamma^{(0)}_0 + \bar{\alpha}_\mathrm{s}\gamma^{(1)}_0 + b\bar{\alpha}_\mathrm{s}\frac{\partial\log\mathfrak{R}(\bar{\alpha}_\mathrm{s};\omega)}{\partial\log\bar{\alpha}_\mathrm{s}}\end{aligned} \tag{5.13}$$

and has therefore subleading terms determined by the ε-dependence of $\bar{\gamma}_\varepsilon$ at $\varepsilon = b\bar{\alpha}_\mathrm{s}$, which are related in the expected way to the coefficient $\mathfrak{R}$.

On the other hand, the gluon density in the Q_0-scheme contains an additional fluctuation factor $\mathcal{N}(\bar\alpha_s;\omega)$ whose calculation at full NLx level proceeds along the lines of section 4 and yields

$$\mathcal{N} = \frac{1}{\gamma_0\sqrt{-\chi_0'}}\left\{1+\frac{b\omega}{24\chi_0'}\left[2\left(\frac{\chi_0''}{\chi_0'}\right)^2-3\left(\frac{\chi_0''}{\chi_0'}\right)'\right]+\bar\alpha_s\left[\frac{\chi_0^{(1)}}{\gamma_0\chi_0'}+\frac{1}{2}\left(\frac{\chi_0^{(1)\prime}\chi_0''}{(\chi_0')^2}-\frac{\chi_0^{(1)\prime\prime}}{\chi_0'}\right)\right]\right\}. \tag{5.14}$$

The corresponding anomalous dimension is

$$\gamma^{(Q_0)}(\bar\alpha_s;\omega) = \gamma_0^{(0)}\left(\frac{\bar\alpha_s}{\omega}\right)+\bar\alpha_s\gamma_0^{(1)}\left(\frac{\bar\alpha_s}{\omega}\right)-b\bar\alpha_s\frac{\partial\log\mathcal{N}(\bar\alpha_s;\omega)}{\partial\log\bar\alpha_s} \tag{5.15}$$

and can be calculated directly at $\varepsilon=0$.

It should be remarked that the NNLx contributions to the anomalous dimension in eqs. (5.13) and (5.15) coming from the NLx corrections to $\mathfrak{R}$ and $\mathcal{N}$ are of course mixed with dynamical contributions whose calculation has not yet been attempted in the literature. Nevertheless, we do compute here — once $\gamma_1^{(1)}$ is known — the full NLx contributions to $\mathfrak{R}$, which therefore explain the *difference* between $\overline{\mathrm{MS}}$- and Q_0-scheme anomalous dimensions at NNLx accuracy.

We end this section by writing down explicitly the difference between the $\overline{\mathrm{MS}}$ and Q_0 anomalous dimensions in NNLx approximation:

$$\gamma^{(\overline{\mathrm{MS}})}-\gamma^{(Q_0)} = b\bar\alpha_s\left[\gamma_1^{(0)}+\bar\alpha_s\gamma_1^{(1)}+b\bar\alpha_s\gamma_2^{(0)}+\frac{\partial\log\mathcal{N}}{\partial\log\bar\alpha_s}\right]. \tag{5.16}$$

6. Universality of $\gamma_{\mathrm{qg}}^{(\overline{\mathrm{MS}})}$ and its resummation formula

So far, we have considered the gluon channel only, and discussed NLx corrections to the normalisation change of the gluon density from the $\overline{\mathrm{MS}}$-scheme to the Q_0-scheme. However physical probes are coupled to quarks, which enter the BFKL framework through a $\boldsymbol{k}$-factorisation kernel $H_\varepsilon^{(p)}$, defining the measuring process at hand (labelled by the superscript p), as shown in figure 1.

In a physical scheme, like the Q_0-scheme, we can just *define* the quark density $q^{(p)}$ by the action of $H_\varepsilon^{(p)}$

$$q_\varepsilon^{(p)}(t) = \alpha_s(t)\int \mathrm{d}^{2+2\varepsilon}\boldsymbol{k}'\; H_\varepsilon^{(p)}\Big(\frac{\boldsymbol{k}^2}{\boldsymbol{k}'^2}\Big)\mathcal{F}_\varepsilon(\boldsymbol{k}')\,, \tag{6.1}$$

($\alpha_s(t)\equiv\alpha_s e^{\varepsilon t}$, see eq. (2.1)) and it is then pretty easy to find a 4-dimensional NLx resummation formula for $\gamma_{\mathrm{qg}}^{(p)}$, as first shown by Catani and Hautmann [6] in the DIS-scheme ($q^{(\mathrm{DIS})}=F_2$).

On the other hand, in the $\overline{\mathrm{MS}}$-scheme one has to disentangle the coefficient part of eq. (6.1) from the anomalous dimension part. The latter contains, by definition, the minimal $1/\varepsilon$ singularities which are not exponentiated in the gluon density in the following

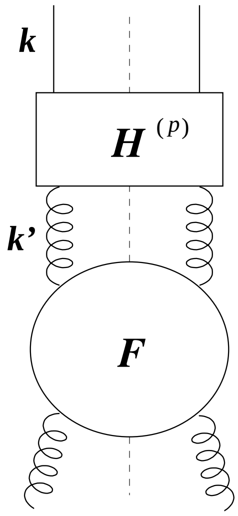

Figure 1: The leading small-x contribution to the quark density.

collinear factorisation formula

$$q_\varepsilon^{(p)}(t) - C_{\mathsf{qg}}^{(p)}\big(\alpha_{\mathrm{s}}(t),\varepsilon\big)\, g_\varepsilon^{(\overline{\mathrm{MS}})}(t) = q_\varepsilon^{(\overline{\mathrm{MS}})}(t) = \int_{-\infty}^{t} \mathrm{d}\tau\; \gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})}\big(\alpha_{\mathrm{s}}(\tau)\big)\, g_\varepsilon^{(\overline{\mathrm{MS}})}(\tau)\,, \tag{6.2}$$

where we have set $C_{\mathsf{qq}}^{(p)} = 1$ and $\gamma_{\mathsf{qq}}^{(\overline{\mathrm{MS}})} = 0$ at NLx anomalous dimension accuracy and we have omitted, for notational simplicity, the ω-variable. Note that $C_{\mathsf{qg}}^{(p)}$ is process- and ε-dependent, while γ_{qg} is universal and ε-independent.

$q^{(p)}$ is obtained in the γ-representation framework by replacing in eq. (6.1) the characteristic function of the $H_\varepsilon^{(p)}$ kernel

$$\int \frac{\mathrm{d}\boldsymbol{k}'^2}{\boldsymbol{k}'^2} \left(\frac{\boldsymbol{k}'^2}{\boldsymbol{k}^2}\right)^{\gamma} H_\varepsilon^{(p)}\Big(\frac{\boldsymbol{k}^2}{\boldsymbol{k}'^2}\Big) \equiv \frac{\mathcal{H}_\varepsilon^{(p)}(\gamma)}{\gamma(\gamma+\varepsilon)}\,, \tag{6.3}$$

where the two poles are expected because of the intermediate $\boldsymbol{l}$-integration in $H_\varepsilon^{(p)}$ ($\boldsymbol{k} \gtrsim \boldsymbol{l} \gtrsim \boldsymbol{k}'$ roughly) and the ε-dimension of the kernel integration providing also a running $\alpha_{\mathrm{s}}(t) = \alpha_{\mathrm{s}} \cdot (\boldsymbol{k}^2/\mu^2)^\varepsilon$

$$\int \frac{\mathrm{d}\boldsymbol{k}'^2}{\boldsymbol{k}'^2} \left(\frac{\boldsymbol{k}'^2}{\boldsymbol{k}^2}\right)^{\gamma} \int \frac{\mathrm{d}^{2+2\varepsilon}\boldsymbol{l}}{\boldsymbol{l}^2}\, \Theta(\boldsymbol{k}^2 - \boldsymbol{l}^2)\Theta(\boldsymbol{l}^2 - \boldsymbol{k}'^2) = c_\varepsilon\, \frac{(\boldsymbol{k}^2)^\varepsilon}{\gamma(\gamma+\varepsilon)}\,. \tag{6.4}$$

The pole $1/\gamma$ in (6.3) is incorporated into the integrated gluon density $g_\varepsilon(\gamma) = f_\varepsilon(\gamma)/\gamma$ to yield

$$q_\varepsilon^{(p)}(t) = \alpha_{\mathrm{s}}(t) \int \mathrm{d}\gamma\; \mathrm{e}^{\gamma t} g_\varepsilon(\gamma) \frac{\mathcal{H}_\varepsilon^{(p)}(\gamma)}{\gamma+\varepsilon}\,, \tag{6.5}$$

where

$$g_\varepsilon(t) = \int \mathrm{d}\gamma\; \mathrm{e}^{\gamma t} g_\varepsilon(\gamma) = R\Big(\frac{\bar{\alpha}_{\mathrm{s}}(t)}{\omega}, \varepsilon\Big)\, g_\varepsilon^{(\overline{\mathrm{MS}})}(t)\,. \tag{6.6}$$

One could try a saddle point evaluation of q around $\gamma_0\big(\bar{\alpha}_\mathrm{s}(t)/\omega\big)$, but it is difficult to get the right accuracy, and the fluctuation algorithm is cumbersome. We prefer to extract γ_{qg} directly from the expression

$$\begin{aligned}
\dot{q}_\varepsilon^{(p)}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} q_\varepsilon^{(p)}(t) &= \alpha_\mathrm{s}(t) \int \mathrm{d}\gamma\, \mathrm{e}^{\gamma t}\, \mathcal{H}_\varepsilon^{(p)}(\gamma) g_\varepsilon(\gamma) && \text{(6.7a)}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big[C_{\mathsf{qg}}^{(p)}\big(\alpha_\mathrm{s}(t),\varepsilon\big) g_\varepsilon^{(\overline{\mathrm{MS}})}(t)\Big] + \gamma_{\mathsf{qg}}\big(\alpha_\mathrm{s}(t)\big) g_\varepsilon^{(\overline{\mathrm{MS}})}(t) && \text{(6.7b)}\\
&= \left\{\left[\varepsilon \frac{\partial}{\partial \log \alpha_\mathrm{s}(t)} + \gamma_0\Big(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\Big)\right] C_{\mathsf{qg}}^{(p)}\big(\alpha_\mathrm{s}(t),\varepsilon\big) + \gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})}\big(\alpha_\mathrm{s}(t)\big)\right\} g_\varepsilon^{(\overline{\mathrm{MS}})}(t)\,, && \text{(6.7c)}
\end{aligned}$$

having used in the first line the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\alpha_\mathrm{s}(t)\mathrm{e}^{\gamma t}\right] = \alpha_\mathrm{s}(t)\mathrm{e}^{\gamma t}(\gamma + \varepsilon)\,. \tag{6.8}$$

It is not obvious how to extract from eq. (6.7) resummation formulas for both $C_{\mathsf{qg}}^{(p)}$ and γ_{qg} in terms of the $\boldsymbol{k}$-factorisation integral of $\dot{q}_\varepsilon^{(p)}$. For instance, at $\varepsilon = 0$ we get the well known [6] $\boldsymbol{k}$-factorisation result

$$\begin{aligned}
\mathcal{H}_0^{(p)}\Big(\gamma_0\Big(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\Big)\Big) R\Big(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega},0\Big) &= C_{\mathsf{qg}}^{(p)}\big(\alpha_\mathrm{s}(t),0\big)\gamma_0\Big(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\Big) + \gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})}\big(\alpha_\mathrm{s}(t)\big)\\
&= \gamma_{\mathsf{qg}}^{(p)}\big(\alpha_\mathrm{s}(t)\big) R\Big(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega},0\Big)\,,
\end{aligned} \tag{6.9}$$

which determines γ_{qg} in the p-scheme, but not $C_{\mathsf{qg}}^{(p)}$ and $\gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})}$ separately. In ref. [6], a double expansion of (6.7b) in both $\alpha_\mathrm{s}(t)$ and ε is used to get half a dozen terms in the $\alpha_\mathrm{s}(t)$-expansion of γ_{qg}, a result further improved in [17].

Here we want to derive $\gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})}\big(\alpha_\mathrm{s}(t)\big)$ to all orders in $\bar{\alpha}_\mathrm{s}(t)/\omega$, by exploiting the property of the coefficient part in (6.7b) of being a total t-derivative of the product of g_ε and a function which is perturbative in both $\alpha_\mathrm{s}(t)$ and ε. We start noticing that $\mathcal{H}_\varepsilon^{(p)}$ can be expanded around $\gamma = -\varepsilon$ in the form

$$\begin{aligned}
\mathcal{H}_\varepsilon^{(p)}(\gamma) &= \mathcal{H}_\varepsilon^{(p)}(-\varepsilon) + \mathcal{O}\left(\gamma+\varepsilon\right)\\
&\equiv \mathcal{H}(\varepsilon) + \mathcal{O}\left(\gamma+\varepsilon\right)\,,
\end{aligned} \tag{6.10}$$

where $\mathcal{H}(\varepsilon)$ is now process-independent, being the residue of the function (6.1) at the collinear pole $\gamma = -\varepsilon$. In appendix B we evaluate $\mathcal{H}_\varepsilon^{(p)}(\gamma)$ in various cases and we show that in all cases $\mathcal{H}(\varepsilon)$ is process-independent and can be written as the product of a rational and a transcendental part

$$\mathcal{H}(\varepsilon) = \left[\frac{T_R}{2\pi}\,\frac{2}{3}\,\frac{1+\varepsilon}{(1+2\varepsilon)(1+\frac{2}{3}\varepsilon)}\right]\left[\frac{\mathrm{e}^{\varepsilon\psi(1)}\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1+2\varepsilon)}\right] \equiv \mathcal{H}^{\mathrm{rat}}(\varepsilon)\mathcal{H}^{\mathrm{tran}}(\varepsilon)\,. \tag{6.11}$$

This form basically follows from the off-shell generalisation of the $\mathsf{g} \to \mathsf{q}\bar{\mathsf{q}}$ DGLAP splitting function introduced in [6], and proved in appendix B.3 to have a universal off-shell dependence induced by $\boldsymbol{k}$-factorisation.

On the other hand, by eq. (6.8) terms of order $(\gamma+\varepsilon)^n$ ($n \geq 1$) are easily seen to be total t-derivatives, so that we can rewrite eq. (6.7) in the form

$$\mathcal{H}(\varepsilon)\alpha_s(t)R\Big(\frac{\bar{\alpha}_s(t)}{\omega},\varepsilon\Big)g_\varepsilon^{(\overline{\mathrm{MS}})}(t) = \gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})}\big(\alpha_s(t)\big)g_\varepsilon^{(\overline{\mathrm{MS}})}(t) + \text{total } t\text{-derivative}\,. \tag{6.12}$$

This result shows that $\gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})}$ is universal, being dependent on the universal functions $\mathcal{H}(\varepsilon)$ and R. Furthermore, it suggests how to extract the anomalous dimension contributions from the power series in ε in the l.h.s., i.e., by subtracting a total t-derivative or, in other words, by doing an "integration by parts". By writing the expansions

$$\mathcal{H}^{\mathrm{tran}}(\varepsilon)R\Big(\frac{\bar{\alpha}_s(t)}{\omega},\varepsilon\Big) = \sum_{n=0}^{\infty}(-\varepsilon)^n R_n\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big) \tag{6.13}$$

$$\mathcal{H}^{\mathrm{rat}}(\varepsilon) = \sum_{m=0}^{\infty}(-\varepsilon)^m \mathcal{H}_m \tag{6.14}$$

the general term of the series is of the form $(-\varepsilon)^{n+m}R_n\big(\bar{\alpha}_s(t)/\omega\big)$ and is reduced to a power series in $\alpha_s(t)$ by repeated application of the identity

$$-\varepsilon\alpha_s(t)\rho\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)g_\varepsilon^{(\overline{\mathrm{MS}})}(t) = \alpha_s(t)\gamma_0\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)\tilde{\rho}\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)g_\varepsilon^{(\overline{\mathrm{MS}})}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\left[\alpha_s(t)\tilde{\rho}\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)g_\varepsilon^{(\overline{\mathrm{MS}})}(t)\right] \tag{6.15}$$

valid for any function ρ, where

$$\tilde{\rho} \equiv \left(1+\frac{\partial}{\partial\log\alpha_s(t)}\right)^{-1}\rho \equiv \left(1+\hat{D}\right)^{-1}\rho\,. \tag{6.16}$$

Eq. (6.15) means that multiplication by $-\varepsilon$ corresponds to the operator $\gamma_0\big(\bar{\alpha}_s(t)/\omega\big)(1+\hat{D})^{-1}$ after integration by parts. It follows that the quark anomalous dimension is given by

$$\gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})}\big(\alpha_s(t);\omega\big) = \alpha_s(t)\mathcal{H}^{\mathrm{rat}}\left(-\gamma_0\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)\frac{1}{1+\hat{D}}\right)\sum_{n=0}^{\infty}\left(\gamma_0\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)\frac{1}{1+\hat{D}}\right)^n R_n\Big(\frac{\bar{\alpha}_s(t)}{\omega}\Big)\,, \tag{6.17}$$

which is the resummed expression we were looking for.

The subtlety of the result (6.17) is that, in order to get a resummed formula in $\bar{\alpha}_s(t)/\omega$ of the ε-independent γ_{qg}, we need an all order ε-expansion of the $\boldsymbol{k}$-factorisation formula (6.12). The fact that terms of order ε^n generate finite contributions to both coefficient and anomalous dimension is somewhat similar to what already noticed in the gluon channel, and is typical of the minimal subtraction recipe. However, while higher orders in ε correspond to higher subleading $\log 1/x$ resummation levels in the gluon case, here they just correspond to higher orders in the $\bar{\alpha}_s(t)/\omega$-expansion of γ_{qg}.

In order to use eq. (6.17) we need to understand the action of the $\gamma_0(1+\hat{D})^{-1}$ operator. A simple example is provided by setting $R_n = \delta_{n0}$ and $\gamma_0 = \bar{\alpha}_s(t)/\omega$. By noting that $\hat{D}[\alpha_s(t)]^n = n[\alpha_s(t)]^n$, we obtain

$$\gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})} = \alpha_s(t)\sum_{n=0}^{\infty}\frac{\mathcal{H}_n}{n!}\left(\frac{\bar{\alpha}_s(t)}{\omega}\right)^n = \alpha_s(t)\mathcal{H}_B^{\mathrm{rat}}\Big(-\frac{\bar{\alpha}_s(t)}{\omega}\Big) = \frac{\alpha_s(t)T_R}{4\pi}\left(\mathrm{e}^{2\frac{\bar{\alpha}_s(t)}{\omega}}+\frac{1}{3}\mathrm{e}^{\frac{2}{3}\frac{\bar{\alpha}_s(t)}{\omega}}\right)\,, \tag{6.18}$$

where $\mathcal{H}_B$ denotes the Borel transform of $\mathcal{H}^{\rm rat}$, which is simply obtained from eq. (6.11). As noticed in [6], eq. (6.18) provides all rational coefficients occurring in the resummed formula for $\gamma_{\sf qg}^{(\overline{\rm MS})}$.

It is also possible to provide an expression for (6.17) involving only quadratures, provided the functions R_n are given to all orders. In appendix D the result is derived in terms of the intermediate function

$$\hat{R}\Big(\frac{\bar{\alpha}_{\rm s}}{\omega}\Big) = \frac{\partial}{\partial(\bar{\alpha}_{\rm s}/\omega)} \int_0^{\bar{\alpha}_{\rm s}/\omega} {\rm d}a \; R_B\Big(a, -\int_a^{\bar{\alpha}_{\rm s}/\omega} \frac{{\rm d}a'}{a'}\gamma_0(a')\Big) \tag{6.19}$$

where R_B is the Borel transform of $\mathcal{H}^{\rm tran}(\varepsilon)R(\bar{\alpha}_{\rm s}/\omega,\varepsilon)$ in the $-\varepsilon$ variable. The final result is

$$\begin{aligned}\gamma_{\sf qg}^{(\overline{\rm MS})} &= \frac{\alpha_{\rm s}(t)}{2\pi}\frac{T_R}{2}\frac{\partial}{\partial(\bar{\alpha}_{\rm s}/\omega)} \int_0^{\bar{\alpha}_{\rm s}/\omega} {\rm d}a \bigg[\exp\left(2\int_a^{\bar{\alpha}_{\rm s}/\omega}\frac{{\rm d}a'}{a'}\,\gamma_0(a')\right) \\ &\qquad + \frac{1}{3}\exp\left(\frac{2}{3}\int_a^{\bar{\alpha}_{\rm s}/\omega}\frac{{\rm d}a'}{a'}\,\gamma_0(a')\right)\bigg]\hat{R}(a)\,, \end{aligned} \tag{6.20}$$

where we notice the exponentials of (6.18) occurring in a more general framework.

7. Discussion

The main results of this paper are the renormalisation group (RG) factorisation of the BFKL equation in $4+2\varepsilon$ dimensions at NLx level, and the relation of the Q_0-scheme to the $\overline{\rm MS}$-scheme at the same level of accuracy. The collinear factorisation has been proved by solving the BFKL equation with a cutoff $Q_0^2 = \mu^2 {\rm e}^{-T}$, and by showing that — in the on-shell limit for the initial gluon ($Q_0 = 0$) — the solution admits a RG representation with a well-defined anomalous dimension $\bar{\gamma}_\varepsilon(\bar{\alpha}_{\rm s};\omega)$, as given by eqs. (3.10), (4.5) and (5.10) at various levels of accuracy. This representation shows exponentiated IR poles $\sim \frac{1}{\varepsilon}(\frac{b\bar{\alpha}_{\rm s}}{\varepsilon})^n$, which are factorised in the IR-free regime with $\bar{\alpha}_{\rm s}(t) < \varepsilon/b$ and $\bar{\alpha}_{\rm s}(-\infty) = 0$. Then, by the ε-expansion of $\bar{\gamma}_\varepsilon(\bar{\alpha}_{\rm s};\omega)$ in eqs. (4.3) and (5.9), we are able to define the minimal subtraction scheme, to switch to the UV-free regime, and to find the transformation factor $R(\bar{\alpha}_{\rm s};\omega)$ at NLx level.

We discover in this way that the coefficient R is due to the product of a fluctuation factor $\mathcal{N}$, which can be calculated at $\varepsilon = 0$, and of a dynamical factor $\mathfrak{R} = R/\mathcal{N}$ which reflects the ε-dependence of $\bar{\gamma}_\varepsilon(\bar{\alpha}_{\rm s};\omega)$, expanded around $\varepsilon = b\bar{\alpha}_{\rm s}$. Furthermore, the ε-dependent $\bar{\gamma}$ directly provides subleading contributions to the $\overline{\rm MS}$-scheme anomalous dimension, encoded in $\bar{\gamma}_{\varepsilon=b\bar{\alpha}_{\rm s}}(\bar{\alpha}_{\rm s};\omega)$ (eqs. (4.15), (5.11) and (5.12)). In this way, the difference $\gamma^{(\overline{\rm MS})} - \gamma^{(Q_0)}$ between the two schemes is here calculated up to NNLx level (eq. (5.16)). Therefore, the ε-dependence of the kernel is transmuted into a subleading $\alpha_{\rm s}$-dependence for the $\overline{\rm MS}$ anomalous dimension.

A similar transmutation phenomenon occurs in the case of quark-gluon mixing. In addition, due to the different form of the scheme-changing transformation (6.2), the ε-dependence induces *leading* $\alpha_{\rm s}/\omega$-dependence into $\gamma_{\sf qg}^{(\overline{\rm MS})}$ — which has to be disentangled from the process-dependent contributions to $C_{\sf qg}^{(p)}$. Therefore, the knowledge of the ε-dependence becomes increasingly important for the full singlet evolution.

The above results for R and $\mathcal{N}$ at NLx level directly provide the $Q_0 \leftrightarrow \overline{\text{MS}}$ scheme change for the gluon anomalous dimension at NNLx level (eq. (5.16)). Although resummed results at this level of accuracy are not yet available, a NNLx term $\sim \bar{\alpha}_s^3/\omega$ is present in the recent NNLO calculation [21] in the $\overline{\text{MS}}$-scheme, and could be translated to the Q_0-scheme[7] by using eq. (5.16).

If, on the other hand, the all-order NNLx level is considered, the problem arises of the small-x behaviour of the scheme changing transformation. Both the LLx and NLx expressions of R show severe leading Pomeron singularities, which have alternating sign and call for a doubly resummed approach in which a much softer singularity is expected. It is not clear, at present, how to apply our new method to such resummed approaches [14, 15]. However, the present results provide some hints towards an improved scheme-changing transformation. First of all, the analysis of section 6 and appendix B shows that there is a universal part in the ε-dependence of $\mathcal{H}_\varepsilon^{(p)}(\gamma)$ (the p-scheme defining kernel) which is encoded in the collinear pole at $\gamma = -\varepsilon$, which in turn comes from an off-shell generalisation of the well-known P_{qg} splitting function. It is then conceivable that a similar analysis can be performed for resummed models as well, in order to control the leading part of the scheme-changing transformation and its mixing.

Furthermore, since the higher order ε-dependence affects R at subleading level, it is conceivable that most of the normalisation change comes from the already known (or generalised) leading part. A preliminary analysis in this direction is under way [19]. Hopefully, this will soon lead to a realistic comparison of the resummed approach [14] to experimental data.

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A. Solutions of the difference equations in γ-space

A.1 Asymptotic solution

In section 2 we showed that the solution of both the homogeneous and inhomogeneous equation (2.8) in γ-space are determined by difference equations whose structure is, in both cases, given by (cf. eqs. (2.13) and (2.28))

$$S_\varepsilon(\gamma + \varepsilon) - S_\varepsilon(\gamma) = L_\varepsilon(\gamma)\,. \tag{A.1}$$

Evidently, eq. (A.1) determines $S_\varepsilon(\gamma)$ up to an additive periodic function of γ of period ε. Here we want to provide a formal solution for S_ε in terms of the known function L_ε by keeping in mind that, by the remark above, the solution is not necessarily single valued. Due to the non-local form in γ of eq. (A.1), we expect S_ε to be determined by an infinite number of γ-derivatives of L_ε. In fact, ∂_γ is the generator of γ-translations, and $\exp(\varepsilon\partial_\gamma)$

[7] We thank Richard Ball and Stefano Forte for a discussion on this point.

the translation operator:

$$\left[\exp(\varepsilon\partial_\gamma)S_\varepsilon\right](\gamma) = S_\varepsilon(\gamma+\varepsilon)\,, \tag{A.2}$$

so that, by substituting eq. (A.2) into eq. (A.1) we obtain the formal solution

$$S_\varepsilon = \left[\exp(\varepsilon\partial_\gamma) - 1\right]^{-1} L_\varepsilon\,. \tag{A.3}$$

Since we are not able to provide the resolvent in eq. (A.3) in closed form, we look for a perturbative solution in ε of S_ε. Given that $\exp(\varepsilon\partial_\gamma) - 1 \sim \varepsilon\partial_\gamma$ for $\varepsilon \to 0$, we can rewrite the r.h.s. of eq. (A.3) in terms of an operator close to the identity by applying $\varepsilon\partial_\gamma$ to the left, obtaining

$$\mathcal{L}_\varepsilon \equiv \varepsilon\partial_\gamma S_\varepsilon = \frac{\varepsilon\partial_\gamma}{\exp(\varepsilon\partial_\gamma)-1} L_\varepsilon = \sum_{n=0}^{\infty} \frac{B_n}{n!}\varepsilon^n L_\varepsilon^{(n)}(\gamma)\,, \tag{A.4}$$

where we have adopted the definition (2.12) and used the generating function (2.15) of the Bernoulli numbers. This proves eq. (2.14).

A remark concerning the character of the Series in eq. (A.4): since the asymptotic behaviour of the Bernoulli numbers is $|B_n| \sim 2\,n!/(2\pi)^n$ (for even n, while $B_n = 0$ for odd $n > 1$) it turns out that the series has a non-vanishing radius of convergence if and only if the series $\sum_n L_\varepsilon^{(n)}(\gamma)x^n/n!$ has infinite radius of convergence, i.e., $L_\varepsilon(\gamma)$ is holomorphic in the whole γ-plane. If $L_\varepsilon(\gamma)$ has at least one singularity at finite values of γ, then the series in eq. (A.4) is an asymptotic series of $\mathcal{L}_\varepsilon(\gamma)$ for $\varepsilon \to 0$.

A straightforward example is provided by the Euler ψ function which satisfies $\psi(z+1) - \psi(z) = 1/z$. By rescaling $z \equiv \gamma/\varepsilon$ and $\psi(z) \equiv S_\varepsilon(\gamma)$, we reproduce eq. (A.1) with $L_\varepsilon(\gamma) = \varepsilon/\gamma$, hence $L_\varepsilon^{(n)}(\gamma) = \varepsilon(-1)^n n!/\gamma^{n+1}$ which, substituted in eq. (A.4) yields $\psi'(z) = \mathcal{L}_\varepsilon(\gamma) = \sum_{n=0}^{\infty} B_n(-1)^n(\varepsilon/\gamma)^{n+1}$ and finally $\psi(z) = \log z + \sum_{n=1}^{\infty} B_n(-1)^{n+1}/(nz^n) + c$, which is exactly the asymptotic expansion of the ψ function for large $|z|$, provided we set the constant of integration $c = 0$. Note that also the function $z \mapsto \psi(1-z)$ satisfies the same difference equation as $\psi(z)$, and in fact they differ by the periodic function $\pi\mathrm{cotg}(\pi z)$. The asymptotic expansion of $\psi(1-z)$ differs from that of $\psi(z)$ by the constant $-\mathrm{i}\pi\mathrm{sign}\big(\Im(z)\big)$ and is therefore still consistent with our series representation. In practice, this kind of ambiguities requires the choice of a Riemann sheet for the asymptotic solution $S_\varepsilon(\gamma)$, which has one or several branch-points. This is done on the basis of the regularity requirements for the BFKL solution in the UV or IR region. For instance, the saddle point (2.36) is dominant in the Riemann sheet relevant for the UV regular solution of type (2.35).

A.2 Iterative solution

The BFKL equation can be solved through the iterative method, both in t- and γ-space. In this section we want to prove the equivalence of these two procedures.

Let us first review the iterative method in t-space. From the integral equation (2.2) for $\mathcal{F}_\varepsilon(\boldsymbol{k})$ we derive the corresponding equation for $\widetilde{\mathcal{F}}_\varepsilon(t) \equiv \widetilde{\mathcal{F}}_\varepsilon(\boldsymbol{k})$:

$$\widetilde{\mathcal{F}}_\varepsilon(t) = \frac{\bar{\alpha}_\mathrm{s}}{\omega}\mathrm{e}^{\varepsilon t}\left[1 + \int \mathrm{d}t'\; K_\varepsilon(t-t')\widetilde{\mathcal{F}}_\varepsilon(t')\right]\,, \tag{A.5}$$

where the rescaled and azimuthally averaged kernel

$$K_\varepsilon(t-t') \equiv \frac{\mathrm{e}^{\varepsilon\psi(1)}}{\Gamma(1+\varepsilon)} \frac{\pi \boldsymbol{k}^2}{g^2 N_c} \left\langle \mathcal{K}_\varepsilon(\boldsymbol{k},\boldsymbol{k}')\right\rangle \equiv \frac{\mathrm{e}^{\varepsilon\psi(1)}\boldsymbol{k}^2}{2\pi^\varepsilon g^2 N_c} \int_{\mathbb{S}_{1+2\varepsilon}} \mathrm{d}^{1+2\varepsilon}\hat{\boldsymbol{k}}' \, \mathcal{K}_\varepsilon(\boldsymbol{k},\boldsymbol{k}') \tag{A.6}$$

is a dimensionless scale-invariant kernel whose eigenfunctions are exponentials $\mathrm{e}^{\gamma t} : 0 < \Re\gamma < 1$, and the corresponding eigenvalue function is

$$\chi_\varepsilon(\gamma) = \mathrm{e}^{-\gamma t} \int \mathrm{d}t' \, K_\varepsilon(t-t')\mathrm{e}^{\gamma t'} \,, \qquad (0 < \Re\gamma < 1) \,. \tag{A.7}$$

The iterative solution of eq. (A.5) is then given by

$$\widetilde{\mathcal{F}}_\varepsilon(t) = \sum_{n=0}^{\infty} \widetilde{\mathcal{F}}_\varepsilon^{[n]}(t) \,, \tag{A.8}$$

where

$$\widetilde{\mathcal{F}}_\varepsilon^{[0]}(t) = \frac{\bar{\alpha}_\mathrm{s}}{\omega}\mathrm{e}^{\varepsilon t} \,, \tag{A.9a}$$

$$\widetilde{\mathcal{F}}_\varepsilon^{[n]}(t) = \frac{\bar{\alpha}_\mathrm{s}}{\omega}\mathrm{e}^{\varepsilon t} \int \mathrm{d}t' \, K_\varepsilon(t-t')\widetilde{\mathcal{F}}_\varepsilon^{[n-1]}(t') = \left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\mathrm{e}^{\varepsilon t}\right)^{n+1} \prod_{k=1}^{n} \chi_\varepsilon(k\varepsilon) \tag{A.9b}$$

so as to yield eq. (2.4). Note however that the action of the kernel K_ε on $\widetilde{\mathcal{F}}_\varepsilon^{[n-1]} \sim \mathrm{e}^{n\varepsilon t}$ in eq. (A.9b) is defined only if $n\varepsilon < 1$. For $n > 1/\varepsilon$ the functions $\widetilde{\mathcal{F}}_\varepsilon^{[n-1]}$ do not belong anymore to the domain of K_ε and the iterative procedure breaks up. Nevertheless, from a perturbative point of view, eq. (2.4) is meaningful, because for any given order n the coefficient of $\bar{\alpha}_\mathrm{s}(t)^{n+1}$ is well-defined and analytic in a non-empty open interval $]0, 1/n[$ of ε-values.

In γ-space, on the other hand, the iterative solution (2.24) has been obtained by using an initial condition with an infrared cutoff T — see Eqs. (2.9,2.10) in comparison with eq. (A.9a) — so as to deal with analytic (non symbolic) functions.[8] In order to prove the equivalence of eq. (2.4) and eq. (2.24), we rewrite the latter in the form

$$f_\varepsilon(\gamma) = \sum_{n=0}^{\infty} f_\varepsilon^{[n]}(\gamma) \,, \tag{A.10}$$

$$f_\varepsilon^{[n]}(\gamma) = \left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)^{n+1} \frac{\mathrm{e}^{[\gamma-(n+1)\varepsilon]T}}{\gamma-(n+1)\varepsilon} \prod_{k=1}^{n} \chi_\varepsilon(\gamma - k\varepsilon) \,, \tag{A.11}$$

where we have used the explicit form (2.9) of $f^{(0)}$. In performing the inverse Fourier transform (2.7), the real part $\Re\gamma = c$ of the integration path must lie to the right of the UV singularities of $f_\varepsilon^{[n]}(\gamma)$ at $\gamma = m\varepsilon : m = 1, \ldots, n+1$ and to the left of the IR ones at $\gamma = 1 + m\varepsilon : m = 1, \ldots, n+1$. Therefore, also in γ-space the iterative solution appears to be defined only for $n < 1/\varepsilon$. Here, however, we can deform the integration path and extend

[8]The Fourier transform of an exponential is a δ function.

the validity of the γ-representation to higher values of n, though this requires crossing the real axis at larger and larger values of $\gamma > n\varepsilon$ and to properly exclude the IR poles of the eigenvalue functions $\chi_\varepsilon(\gamma - m\varepsilon)$ for $\gamma - m\varepsilon > 1$.

We thus calculate, for $n < 1/\varepsilon$ and $(1+n)\varepsilon < c < 1+\varepsilon$

$$\widetilde{\mathcal{F}}_\varepsilon^{[n]}(t) = \int_{\Re\gamma=c} \frac{\mathrm{d}\gamma}{2\pi\mathrm{i}}\, \mathrm{e}^{\gamma t} f_\varepsilon^{[n]}(\gamma) = \left(\frac{\bar{\alpha}_\mathrm{s}}{\omega}\right)^{n+1} \mathrm{e}^{-(n+1)\varepsilon T} \int_{\Re\gamma=c} \frac{\mathrm{d}\gamma}{2\pi\mathrm{i}}\, \frac{\mathrm{e}^{\gamma(t+T)}}{\gamma-(n+1)\varepsilon} \prod_{k=1}^{n} \chi_\varepsilon(\gamma - k\varepsilon)\,. \tag{A.12}$$

For $t > -T$ we can close the contour path to the left, and pick up the rightmost pole of $f^{(0)}$ at $\gamma = (n+1)\varepsilon$ and other poles of the χ_ε's at $\gamma = m\varepsilon : m < n+1$. Each pole contributes with a $t-$ and T-dependent exponential of the form $\exp[-(n+1-m)\varepsilon T + m\varepsilon t] : m \leq n+1$. Therefore, the residue of the rightmost pole ($m = n+1$) is T-independent, while the other residues are suppressed as integer powers of $\exp(-\varepsilon T)$ and vanish in the $T \to +\infty$ limit. In conclusion

$$\widetilde{\mathcal{F}}_\varepsilon^{[n]}(t) = \left(\frac{\bar{\alpha}_\mathrm{s}(t)}{\omega}\right)^{n+1} \prod_{k=1}^{n} \chi_\varepsilon\big((n+1-k)\varepsilon\big) + \mathcal{O}\left(\mathrm{e}^{-\varepsilon T}\right) \tag{A.13}$$

and, in the $T \to +\infty$ limit we reproduce the t-space iteration (A.9b).

A.3 Calculation of the ρ-function

The calculation of the function $\rho_\varepsilon(\gamma)$ obeying eq. (2.28) requires a special care with respect to the calculation of $S_\varepsilon(\gamma)$ in eq. (2.13), even at leading ε accuracy, because the r.h.s. of eq. (2.13) is finite for $\varepsilon \to 0$, while that of eq. (2.28) has an essential singularity. By using the expression (2.12) for $h_\varepsilon(\gamma)$ into the r.h.s. of eq. (2.28), according to eq. (2.14) we expect

$$\varepsilon\partial_\gamma\rho_\varepsilon(\gamma) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \varepsilon^n \partial_\gamma^n \left[f^{(0)}(\gamma) \mathrm{e}^{-L_\varepsilon(\gamma)} \mathrm{e}^{-\frac{1}{\varepsilon}\int^\gamma \mathcal{L}_\varepsilon(\gamma')\mathrm{d}\gamma'} \right]\,. \tag{A.14}$$

In computing the logarithmic γ-derivative of the product in square brackets, the large contributions are represented by T (from the exponent of $f^{(0)}$, cf. eq. (2.9)), and by the derivative of the "action", which is of $\mathcal{O}(1/\varepsilon)$. The other contributions are T-independent and regular in ε, so that for higher derivatives one gets

$$\begin{aligned} &\partial_\gamma^n \left[f^{(0)}(\gamma) \mathrm{e}^{-L_\varepsilon(\gamma)} \mathrm{e}^{-\frac{1}{\varepsilon}\int^\gamma \mathcal{L}_\varepsilon(\gamma')\mathrm{d}\gamma'} \right] = \\ &\qquad = f^{(0)}(\gamma) \mathrm{e}^{-L_\varepsilon(\gamma)} \mathrm{e}^{-\frac{1}{\varepsilon}\int^\gamma \mathcal{L}_\varepsilon(\gamma')\mathrm{d}\gamma'} \left(T - \frac{1}{\varepsilon}\mathcal{L}_\varepsilon(\gamma) \right)^n \times \big[1 + \mathcal{O}(\varepsilon) \big]\,. \end{aligned} \tag{A.15}$$

Eq. (A.14) becomes then

$$\begin{aligned} \varepsilon\partial_\gamma\rho_\varepsilon(\gamma) &= f^{(0)}(\gamma) \mathrm{e}^{-L_0(\gamma)} \mathrm{e}^{-S_\varepsilon(\gamma)} \sum_{n=0}^{\infty} \frac{B_n}{n!} \left[\varepsilon T - L_0(\gamma) \right]^n \times \big[1 + \mathcal{O}(\varepsilon) \big] \\ &= f^{(0)}(\gamma) \mathrm{e}^{-L_0(\gamma)} \mathrm{e}^{-S_\varepsilon(\gamma)} \frac{\varepsilon T - L_0(\gamma)}{\mathrm{e}^{\varepsilon T - L_0(\gamma)} - 1} \times \big[1 + \mathcal{O}(\varepsilon) \big]\,, \end{aligned} \tag{A.16}$$

where in the last equality use has been made of eq. (2.15). Finally, after integration in γ, one ends up with eq. (2.29).

B. Characteristic functions of the quark kernels $H_\varepsilon^{(p)}$

In this section we want to compute the characteristic functions of the $\mathsf{g} \to \mathsf{q}$ kernels used to define the $\overline{\mathrm{MS}}$ and Q_0 quark densities in the high-energy regime according to eq. (6.1). These kernels are naturally provided in $(x, \boldsymbol{k})$-space, and are related to their ω-moments in eqs. (6.1), (6.3) through a standard Mellin transform. In the collinear limit, they turn out to be all determined by the same off-shell generalisation of the P_{qg} splitting function.

B.1 The Catani-Hautmann kernel $H^{(\mathrm{CH})}$

The $\mathsf{g} \to \mathsf{q}$ kernel used to define the quark density in the high-energy regime is given in eq. (4.8,4.9,C.6) of Ref. [6] (we call ξ their variable z):

$$\alpha_\mathrm{s}(t) H_\varepsilon^{(\mathrm{CH})}\Big(\xi, \frac{Q^2}{\boldsymbol{k}^2}\Big) = \frac{1}{\xi} \hat{K}_{\mathsf{qg}}\Big(\xi, \frac{\boldsymbol{k}^2}{Q^2}\Big)\Big|_{\text{Ref. [6]}} \tag{B.1}$$

where $t = \log Q^2/\mu^2$ along this section. We want to compute the double Mellin transform (cf. eq. (6.3))

$$\mathcal{H}_\varepsilon^{(\mathrm{CH})}(\omega, \gamma) \equiv \gamma(\gamma + \varepsilon) \int_0^1 \mathrm{d}\xi\, \xi^\omega \int \frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2} \left(\frac{\boldsymbol{k}^2}{Q^2}\right)^\gamma H_\varepsilon^{(\mathrm{CH})}\Big(\xi, \frac{Q^2}{\boldsymbol{k}^2}\Big) . \tag{B.2}$$

The result at $\omega = 0$ can be read from [6, eq. (C.9)] and in our notations is

$$\mathcal{H}_\varepsilon^{(\mathrm{CH})}(\gamma) \equiv \mathcal{H}_\varepsilon^{(\mathrm{CH})}(0, \gamma) = \frac{T_R}{2\pi}(1+\varepsilon)(4 - 3\gamma + \varepsilon) \frac{\mathrm{e}^{\varepsilon\psi(1)}\Gamma(1+\varepsilon)\Gamma(1+\gamma)\Gamma(1-\gamma)}{\Gamma(1+\gamma+\varepsilon)\Gamma(4-\gamma+\varepsilon)} . \tag{B.3}$$

The coefficient of the collinear pole at $\gamma = -\varepsilon$ is

$$\mathcal{H}(\varepsilon) \equiv \mathcal{H}_\varepsilon^{(\mathrm{CH})}(-\varepsilon) = \frac{T_R}{2\pi} \frac{2}{3} \frac{1+\varepsilon}{(1+2\varepsilon)(1+\frac{2}{3}\varepsilon)} \frac{\mathrm{e}^{\varepsilon\psi(1)}\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1+2\varepsilon)} . \tag{B.4}$$

B.2 The transverse kernel $H^{(T)}$

We define the Q_0 quark density as the high-energy limit of the quark density in the DIS-scheme, i.e., $q^{(Q_0)} \equiv F_2|_{\text{high energy}}$, the corresponding process being depicted in Fig. 2. We choose a reference frame in which the incoming momenta q and p define the longitudinal plane. Then, in the high energy limit, the momenta entering the blob can be decomposed as follows:

$$q = n - xp ; \qquad k = zp + \boldsymbol{k} , \tag{B.5}$$

where n and p are light-like momenta spanning the longitudinal plane orthogonal to the transverse plane. In the latter we use the euclidean representation $\boldsymbol{k}_\mu \boldsymbol{k}^\mu \equiv -\boldsymbol{k}^2 < 0$. The remaining momenta have the following Sudakov decomposition:

$$l = p_{\mathsf{q}} - q = k - p_{\bar{\mathsf{q}}} = -\beta n + \alpha p + \boldsymbol{l} , \tag{B.6a}$$

$$p_{\mathsf{q}} = q + l = (1-\beta)n + (\alpha - x)p + \boldsymbol{l} , \tag{B.6b}$$

$$p_{\bar{\mathsf{q}}} = k - l = \beta n + (z - \alpha)p + \boldsymbol{k} - \boldsymbol{l} . \tag{B.6c}$$

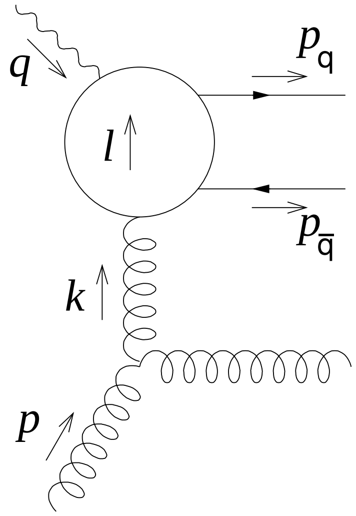

Figure 2: The process $\gamma^* \mathsf{g} \to \mathsf{q}\bar{\mathsf{q}}\mathsf{g}$ used for computing $\mathcal{H}_\varepsilon^{(T)}$.

We then decompose the structure function F_2 in transverse and longitudinal part, each of which is given the factorisation formula (6.1) with an appropriate kernel

$$\alpha_\mathrm{s}(t) H_\varepsilon^{(i)}\Big(\xi, \frac{Q^2}{\boldsymbol{k}^2}\Big) = \frac{g^2 T_R}{(2\pi)^{3+2\varepsilon}} \frac{Q^2}{\xi \boldsymbol{k}^2} \int_0^1 \mathrm{d}\beta \int \mathrm{d}^{2+2\varepsilon}\boldsymbol{l} \times \\ \times \delta\left[1 - \xi\left(1 + \frac{\boldsymbol{l}^2}{(1-\beta)Q^2} + \frac{(\boldsymbol{k}-\boldsymbol{l})^2}{\beta Q^2}\right)\right] C_i(\beta, Q, \boldsymbol{l}, \boldsymbol{k}), \quad (i = L, T), \tag{B.7}$$

where $\xi \equiv x/z$ and

$$C_L = 8\beta^2(1-\beta)^2 Q^2 \left[\frac{1}{D(\boldsymbol{l})} - \frac{1}{D(\boldsymbol{k}-\boldsymbol{l})}\right]^2 \tag{B.8}$$

$$C_T = P_{\mathsf{q}\gamma}(\beta) \left[\frac{\boldsymbol{l}}{D(\boldsymbol{l})} - \frac{\boldsymbol{l}-\boldsymbol{k}}{D(\boldsymbol{l}-\boldsymbol{k})}\right]^2 \tag{B.9}$$

$$D(\boldsymbol{l}) = \boldsymbol{l}^2 + \beta(1-\beta)Q^2\,; \quad P_{\mathsf{q}\gamma}(\beta) = \frac{\beta^2 + (1-\beta)^2 + \varepsilon}{1+\varepsilon}\,. \tag{B.10}$$

The Mellin transform are again computed according to the definition (B.2). The first integration in ξ is easily done by exploiting the δ-function:

$$\alpha_\mathrm{s}(t)\mathcal{H}_\varepsilon^{(i)}(\omega, \gamma) = \frac{g^2 T_R \gamma(\gamma+\varepsilon)}{(2\pi)^{3+2\varepsilon}} \times \\ \times \int \mathrm{d}\beta\, \mathrm{d}^{2+2\varepsilon}\boldsymbol{l}\, \frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2} \left(\frac{\boldsymbol{k}^2}{Q^2}\right)^{\gamma-1} \left(1 + \frac{\boldsymbol{l}^2}{(1-\beta)Q^2} + \frac{(\boldsymbol{k}-\boldsymbol{l})^2}{\beta Q^2}\right)^{-\omega} C_i\,. \tag{B.11}$$

The case $\varepsilon = 0$ has been computed exactly in Ref. [20]:

$$\mathcal{H}_0^{(i)}(\omega, \gamma) = \frac{T_R \gamma^2}{2\pi^2 \alpha_\mathrm{em} \alpha_\mathrm{s}}\; S_i(N = \omega, \gamma)\big|_{\text{Ref. [20]}}\,. \tag{B.12}$$

At $\varepsilon \neq 0$ we are mainly interested in the $\omega = 0$ expression of the transverse kernel, which provides the singular anomalous dimension behaviour

$$\begin{aligned}\alpha_s(t)\mathcal{H}_\varepsilon^{(T)}(\gamma) &\equiv \alpha_s(t)\mathcal{H}_\varepsilon^{(T)}(0,\gamma) \\ &= \frac{g^2 T_R \gamma(\gamma+\varepsilon)}{(2\pi)^{3+2\varepsilon}} \int_0^1 \mathrm{d}\beta \, P_{\mathsf{q}\gamma}(\beta) \int \frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2} \left(\frac{\boldsymbol{k}^2}{Q^2}\right)^{\gamma-1} \int \mathrm{d}\boldsymbol{l} \, \left[\frac{\boldsymbol{l}}{D(\boldsymbol{l})} - \frac{\boldsymbol{l}-\boldsymbol{k}}{D(\boldsymbol{l}-\boldsymbol{k})}\right]^2 .\end{aligned} \tag{B.13}$$

The $\boldsymbol{l}$ integral can be performed by expanding the square, shifting integration variable $\boldsymbol{l} - \boldsymbol{k} \mapsto \boldsymbol{l}$ in the square of the second term inside the brackets, collecting all terms into a single fraction and using Feynman's parameterisation of the common denominator

$$\frac{1}{D^2(\boldsymbol{l})D(\boldsymbol{l}-\boldsymbol{k})} = \int_0^1 \mathrm{d}\lambda \, \frac{2(1-\lambda)}{(\boldsymbol{p}^2+M^2)^3} , \quad \boldsymbol{p} \equiv \boldsymbol{l} - \lambda\boldsymbol{k} , \quad M^2 \equiv \beta(1-\beta)Q^2 + \lambda(1-\lambda)\boldsymbol{k}^2 . \tag{B.14}$$

The corresponding numerator expressed in terms of the new variables reads

$$(1-\lambda)\boldsymbol{k}^2\boldsymbol{p}^2 - 2\lambda(\boldsymbol{k}\cdot\boldsymbol{p})^2 + \lambda\boldsymbol{k}^2M^2 + \text{ odd terms in } \boldsymbol{p} . \tag{B.15}$$

Then, by means of standard integrals in $2 + 2\varepsilon$ dimensions, one can perform the $\boldsymbol{p}$ integrals obtaining additional β- λ- and $\boldsymbol{k}^2$-dependent factors. One can now perform the $\boldsymbol{k}^2$ integration

$$\int \frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2} \left(\frac{\boldsymbol{k}^2}{Q^2}\right)^{\gamma-1} \frac{\boldsymbol{k}^2}{M^{2(1-\varepsilon)}} = Q^{2\varepsilon} B(\gamma, 1-\gamma-\varepsilon) \frac{[\beta(1-\beta)]^{\gamma+\varepsilon-1}}{[\lambda(1-\lambda)]^\gamma} , \tag{B.16}$$

which provides, among other things, the ε-running of the coupling. Then we evaluate the λ integral

$$\int_0^1 \mathrm{d}\lambda \; (1-\lambda)[(1+\varepsilon)(1-\lambda) - \varepsilon\lambda][\lambda(1-\lambda)]^{-\gamma} = \frac{\Gamma^2(1-\gamma)}{\Gamma(2-2\gamma)} \frac{2+\varepsilon-\gamma}{2(3-2\gamma)} \tag{B.17}$$

and the β integral

$$\int_0^1 \mathrm{d}\beta P_{\mathsf{q}\gamma}(\beta)[\beta(1-\beta)]^{\gamma+\varepsilon-1} = \frac{\Gamma(1+\gamma+\varepsilon)}{\Gamma(2+2\gamma+2\varepsilon)} \frac{2[\gamma(1+2\varepsilon)+1+2\varepsilon+2\varepsilon^2]}{(1+\varepsilon)(\gamma+\varepsilon)} \tag{B.18}$$

obtaining the final result

$$\begin{aligned}\mathcal{H}_\varepsilon^{(T)}(\gamma) = \frac{T_R}{\pi} & \frac{\mathrm{e}^{\varepsilon\psi(1)}\Gamma(1+\gamma)\Gamma^2(1-\gamma)\Gamma^2(1+\gamma+\varepsilon)\Gamma(1-\gamma-\varepsilon)}{\Gamma(2-2\gamma)\Gamma(2+2\gamma+2\varepsilon)} \times \\ & \times \frac{(2-\gamma+\varepsilon)[\gamma(1+2\varepsilon)+1+2\varepsilon+2\varepsilon^2]}{(1+\varepsilon)(3-2\gamma)} .\end{aligned} \tag{B.19}$$

The coefficient of the collinear pole of $\mathcal{H}_\varepsilon^{(T)}(\gamma)$ at $\gamma = -\varepsilon$ equals twice the analogous quantity $\mathcal{H}(\varepsilon)$ of eq. (B.4) obtained from the Catani-Hautmann kernel, because here we consider both the quark and antiquark contributions in the fermion loop, while in the kernel $\hat{K}_{\mathsf{qg}}$ given in (B.1) only the quark (or antiquark) contributes.

B.3 Universal collinear behaviour and off-shell splitting function

The fact that the residue at $\gamma = -\varepsilon$ of $\mathcal{H}_\varepsilon^{(p)}(\gamma)/\gamma(\gamma+\varepsilon)$ is independent of p is a simple consequence of the universal collinear behaviour of $H_\varepsilon^{(p)}$ in the region $\boldsymbol{l}^2 \sim \boldsymbol{k}^2 \ll Q^2$, when either l or $k-l$ are collinear with k. In this region we can solve the δ function in eq. (B.7) for small values of β, namely

$$\beta = \frac{\xi}{1-\xi}\frac{(\boldsymbol{k}-\boldsymbol{l})^2}{Q^2} \ll 1\,, \tag{B.20}$$

or instead in the symmetrical region of $1-\beta$ small, with $\boldsymbol{l}$ replaced by $\boldsymbol{k}-\boldsymbol{l}$. Taking the region (B.20), which corresponds to a collinear quark, we see that the longitudinal part is power suppressed, so that it plays no role in the following, and does not contribute to the $\gamma = -\varepsilon$ pole. Furthermore, we can set $P_{\mathsf{q}\gamma}(\beta) = 1$ in the transverse part, with the denominators

$$(1-\xi)D(\boldsymbol{l}) \simeq (1-\xi)\boldsymbol{l}^2 + \xi(\boldsymbol{k}-\boldsymbol{l})^2 = \tilde{\boldsymbol{l}}^2 + \xi(1-\xi)\boldsymbol{k}^2 \equiv \tilde{D} \tag{B.21a}$$

$$(1-\xi)D(\boldsymbol{l}-\boldsymbol{k}) \simeq (\boldsymbol{k}-\boldsymbol{l})^2\,, \qquad (\tilde{\boldsymbol{l}} \equiv \boldsymbol{l} - \xi\boldsymbol{k})\,. \tag{B.21b}$$

By replacing (B.20) and (B.21) in (B.7) we obtain

$$\alpha_s(t)H_\varepsilon^{(T)}\Big(\xi, \frac{Q^2}{\boldsymbol{k}^2}\Big) = \frac{g^2 T_R}{(2\pi)^{3+2\varepsilon}\boldsymbol{k}^2}\int \mathrm{d}^{2+2\varepsilon}\tilde{\boldsymbol{l}}\; \frac{\boldsymbol{l}^2(\boldsymbol{k}-\boldsymbol{l})^2 + 2\boldsymbol{l}\cdot(\boldsymbol{k}-\boldsymbol{l})\tilde{D} + \tilde{D}^2}{\tilde{D}^2}\,. \tag{B.22}$$

Introducing the boost invariant transverse momentum $\tilde{\boldsymbol{l}} \equiv \boldsymbol{l} - \xi\boldsymbol{k}$, the numerator in eq. (B.22) can be recast in the form

$$\tilde{\boldsymbol{l}}^2\boldsymbol{k}^2 - 4\xi(1-\xi)(\boldsymbol{k}\cdot\tilde{\boldsymbol{l}})^2 + 4\xi^2(1-\xi)^2(\boldsymbol{k}^2)^2 + 4\xi(1-\xi)(1-2\xi)\boldsymbol{k}^2(\boldsymbol{k}\cdot\tilde{\boldsymbol{l}}) \tag{B.23}$$

which, after azimuthal averaging in $\tilde{\boldsymbol{l}}$ at fixed $\tilde{\boldsymbol{l}}^2$ and $\boldsymbol{k}^2$, yields ($\tilde{l}^2_{\max} = \lambda(\xi)Q^2$)

$$H_\varepsilon^{(T)}\Big(\xi, \frac{Q^2}{\boldsymbol{k}^2}\Big) \simeq \frac{\mathrm{e}^{\varepsilon\psi(1)}}{2\pi\Gamma(1+\varepsilon)}\int_0^{\lambda(\xi)Q^2}\frac{\mathrm{d}\tilde{\boldsymbol{l}}^2}{\tilde{\boldsymbol{l}}^2}\left(\frac{\tilde{\boldsymbol{l}}^2}{Q^2}\right)^{\varepsilon}\hat{P}^{(0)}_{\mathsf{qg}}\Big(\xi, \frac{\boldsymbol{k}^2}{\tilde{\boldsymbol{l}}^2}, \varepsilon\Big)\,, \quad (\boldsymbol{l}^2, (\boldsymbol{k}-\boldsymbol{l})^2 \ll Q^2)\,. \tag{B.24}$$

This behaviour is identical to that of the Catani-Hautmann kernel $H_\varepsilon^{(\mathrm{CH})}$, with the transverse momentum dependent splitting function

$$\hat{P}^{(0)}_{\mathsf{qg}}(\xi, \kappa^2, \varepsilon) = T_R\left(\frac{1}{1+\xi(1-\xi)\kappa^2}\right)^2\left[\frac{\xi^2+(1-\xi)^2+\varepsilon}{1+\varepsilon} + 4\xi^2(1-\xi)^2\kappa^2\right]\,, \quad \left(\kappa^2 \equiv \frac{\boldsymbol{k}^2}{\tilde{\boldsymbol{l}}^2}\right)\,, \tag{B.25}$$

which thereby acquires a universal meaning. The corresponding contribution to $\mathcal{H}_\varepsilon^{(p)}(\omega,\gamma)$ becomes

$$\begin{aligned}\left.\frac{\mathcal{H}_\varepsilon^{(p)}(\omega,\gamma)}{\gamma(\gamma+\varepsilon)}\right|_{\mathrm{coll}} &= \int_0^1 \mathrm{d}\xi\,\xi^\omega\int_0^\infty\frac{\mathrm{d}\boldsymbol{k}^2}{\boldsymbol{k}^2}\left(\frac{\boldsymbol{k}^2}{Q^2}\right)^\gamma H_\varepsilon^{(T)}\Big(\xi, \frac{Q^2}{\boldsymbol{k}^2}\Big)\\ &= \frac{\mathrm{e}^{\varepsilon\psi(1)}}{2\pi\Gamma(1+\varepsilon)}\int_0^1\mathrm{d}\xi\xi^\omega\int_0^{\lambda(\xi)Q^2}\frac{\mathrm{d}\tilde{\boldsymbol{l}}^2}{\tilde{\boldsymbol{l}}^2}\left(\frac{\tilde{\boldsymbol{l}}^2}{Q^2}\right)^{\gamma+\varepsilon}\int_0^\infty\frac{\mathrm{d}\kappa^2}{\kappa^2}\,(\kappa^2)^\gamma\hat{P}^{(0)}_{\mathsf{qg}}(\xi,\kappa^2,\varepsilon)\end{aligned}$$

$$\overset{\gamma \simeq -\varepsilon}{\simeq} \frac{1}{\gamma+\varepsilon} \frac{\mathrm{e}^{\varepsilon\psi(1)}}{2\pi\Gamma(1+\varepsilon)} \int_0^1 \mathrm{d}\xi\ \xi^\omega \left[\int_0^\infty \frac{\mathrm{d}\kappa^2}{\kappa^2}\ (\kappa^2)^\gamma \hat{P}^{(0)}_{\mathsf{qg}}(\xi,\kappa^2,\varepsilon)\right]_{\gamma=-\varepsilon} . \tag{B.26}$$

We see that the residue at the $\gamma = -\varepsilon$ pole is independent of the details of the $\tilde{\boldsymbol{l}}$ phase space $\tilde{\boldsymbol{l}}^2 < \lambda(\xi)Q^2$ and is given by a simple (analytically continued) moment of the $\hat{P}^{(0)}_{\mathsf{qg}}$ splitting function. We obtain, for $\omega = 0$, the explicit expression

$$\mathcal{H}(\varepsilon) = \frac{\mathrm{e}^{\varepsilon\psi(1)}}{2\pi\Gamma(1+\varepsilon)} \int_0^1 \mathrm{d}\xi \int_0^\infty \mathrm{d}\kappa^2\ (\kappa^2)^{-\varepsilon} \left[-\frac{\partial}{\partial\kappa^2}\hat{P}^{(0)}_{\mathsf{qg}}(\xi,\kappa^2,\varepsilon)\right] . \tag{B.27}$$

$$= \frac{T_R}{2\pi}\,\frac{2}{3}\,\frac{1+\varepsilon}{(1+2\varepsilon)(1+\frac{2}{3}\varepsilon)}\,\frac{\mathrm{e}^{\varepsilon\psi(1)}\Gamma^2(1+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1+2\varepsilon)} . \tag{B.28}$$

The above analysis shows that, by combining $\boldsymbol{k}$-factorisation with the collinear behaviour it is possible to define an off-shell splitting function which is process-independent. The residue at the pole $\gamma = -\varepsilon$ is a particular moment of such splitting function.

C. $\mathcal{O}(\varepsilon)$-corrections to the gluon density

In order to compute the corrections of relative order $b\bar{\alpha}_\mathrm{s}$ to $\mathfrak{R}$ and to $\gamma^{(\overline{\mathrm{MS}})}$ we have to calculate the corrections of order $\mathcal{O}(\varepsilon)$ — i.e., of relative order $\mathcal{O}(\varepsilon^2)$ — to the exponent of the solution (3.6) of the (unintegrated) gluon density $\widetilde{\mathcal{F}}_\varepsilon$.

C.1 Calculation of the saddle point fluctuations

These corrections can be computed by means of a saddle point expansion of the γ-representation

$$\widetilde{\mathcal{F}}_\varepsilon(t) = M_\varepsilon \int \mathrm{d}\gamma\ \mathrm{e}^{\gamma t + S_\varepsilon(\gamma)} . \tag{C.1}$$

Here the "action" S_ε has to be expanded to $\mathcal{O}(\varepsilon)$ as done in eq. (2.17), and the normalisation

$$M_\varepsilon = \sqrt{\frac{\bar{\alpha}_\mathrm{s}}{2\pi\varepsilon\omega}} \tag{C.2}$$

can be inferred from eq. (2.35), which also sets to 0 the lower bound of the integral in the action. We split the exponent of eq. (C.1) into an ε-regular part

$$\tilde{E}_\varepsilon(\gamma) \equiv -\frac{1}{2}L_\varepsilon(\gamma) + \frac{\varepsilon}{12}L'_\varepsilon(\gamma) + \mathcal{O}(\varepsilon^2) \tag{C.3}$$

plus a leading-ε phase

$$E_\varepsilon(\gamma) \equiv \gamma t + \frac{1}{\varepsilon}\int_0^\gamma \mathrm{d}\gamma'\ L_\varepsilon(\gamma') , \tag{C.4}$$

whose stationarity condition determines the saddle point $\gamma = \bar{\gamma}_\varepsilon$ (cf. eq. (4.2)), and the size of the fluctuations

$$\sigma_\gamma^2 \equiv -\frac{1}{E''(\bar{\gamma}_\varepsilon)} = \frac{\varepsilon}{-L'_\varepsilon(\bar{\gamma}_\varepsilon)} . \tag{C.5}$$

By expanding around the saddle point and factoring out of the integral the value of the integrand at the saddle point, we can write

$$\widetilde{\mathcal{F}}_\varepsilon(t) = M_\varepsilon \mathrm{e}^{\bar{\gamma}_\varepsilon t + S_\varepsilon(\bar{\gamma}_\varepsilon)} \sqrt{2\pi}\sigma_\gamma \left\langle \exp\left[\sum_{k=3}^{\infty} \frac{E^{(k)}(\bar{\gamma}_\varepsilon)}{k!}(\gamma-\bar{\gamma}_\varepsilon)^k + \sum_{k=1}^{\infty} \frac{\tilde{E}^{(k)}(\bar{\gamma}_\varepsilon)}{k!}(\gamma-\bar{\gamma}_\varepsilon)^k\right]\right\rangle , \tag{C.6}$$

where the "average" $\langle\cdots\rangle$ is defined by

$$\langle f(\gamma)\rangle \equiv \int \frac{\mathrm{d}\gamma}{\sqrt{2\pi}\sigma_\gamma}\, \exp\left[-\frac{(\gamma-\bar{\gamma}_\varepsilon)^2}{2\sigma_\gamma^2}\right] f(\gamma)\,. \tag{C.7}$$

In eq. (C.6) the sum of the $E^{(k)}$ starts from third order because the zeroth order has been taken out of the integral, the first order is zero by definition of the saddle point condition, and the second order provides the gaussian measure in the "average integral" (C.7).

The product out of the "average" in eq. (C.6) — by using the expressions (C.2,C.5) for M_ε and σ_γ and Eqs. (2.17,3.7,4.6) for the exponential — yields

$$M_\varepsilon\, \mathrm{e}^{\bar{\gamma}_\varepsilon t + S_\varepsilon(\bar{\gamma}_\varepsilon)} \sqrt{2\pi}\sigma_\gamma = \frac{1}{\sqrt{-\chi_\varepsilon(\bar{\gamma}_\varepsilon)}} \exp\left[\int_{-\infty}^{t} \mathrm{d}\tau\; \bar{\gamma}_\varepsilon(\tau)\right] \mathrm{e}^{\varepsilon \frac{1}{12} L'_\varepsilon(\bar{\gamma}_\varepsilon)}\,, \tag{C.8}$$

i.e., the gluon density of eq. (3.9) times a correction factor

$$\mathrm{e}^{\varepsilon \frac{1}{12} L'_\varepsilon(\bar{\gamma}_\varepsilon)} = 1 + \varepsilon \frac{1}{12} L'_\varepsilon(\bar{\gamma}_\varepsilon) + \mathcal{O}\left(\varepsilon^2\right)\,. \tag{C.9}$$

The remaining corrections are found from the "average integral". Note first that the "average" of powers of $\gamma - \bar{\gamma}_\varepsilon$ vanishes for odd powers, and is given by

$$\left\langle(\gamma-\bar{\gamma}_\varepsilon)^{2n}\right\rangle = (2n-1)!!\,\sigma_\gamma^{2n} = \varepsilon^n \frac{(2n-1)!!}{\left[-L'_\varepsilon(\bar{\gamma}_\varepsilon)\right]^n} \tag{C.10}$$

for even powers, thus providing higher and higher orders in ε as n increases. However, the coefficients of the saddle point expansion present ε-singularities, due to the leading phase derivatives $E^{(k)} \sim 1/\varepsilon$. When one expands the exponential inside the "average", the m-th order (which is of $\mathcal{O}(1/\varepsilon^m)$) contains powers $(\gamma-\bar{\gamma}_\varepsilon)^j$ only for $j \geq 3m$. Therefore the power in ε of the fluctuations (C.10) increases faster than the inverse power in ε of the order of expansion of the exponential. In practice, if one is interested in computing the "average" to, say, l-th order in ε, it suffices to expand the exponential up to order $m \leq 2l$ and to take terms in the sum up to $k \leq 2l+3-m$. In the present case, we are interested to the first ε-corrections, namely $l = 1$, and we need therefore the first order expansion of the exponential up to $k = 4$ and the second order up to $k = 3$. By denoting $\delta \equiv (\gamma - \bar{\gamma}_\varepsilon)$ we have

$$\begin{aligned}\langle\cdots\rangle_{\mathrm{C.6}} &= \left\langle 1 + \frac{\tilde{E}^{(2)}}{2!}\delta^2 + \frac{E^{(4)}}{4!}\delta^4 + \frac{1}{2}\left(\frac{\tilde{E}^{(1)}}{1!}\delta\right)^2 + \frac{E^{(1)}}{1!}\frac{\tilde{E}^{(3)}}{3!}\delta^4 + \frac{1}{2}\left(\frac{E^{(3)}}{3!}\delta^3\right)^2\right\rangle + \mathcal{O}\left(\varepsilon^2\right) \\ &= 1 + \varepsilon\left[\frac{1}{8}(-L') + \frac{5}{24}\frac{L''^2}{(-L')^3} + \frac{1}{8}\frac{L'''}{(-L')^2}\right] + \mathcal{O}\left(\varepsilon^2\right)\,,\end{aligned} \tag{C.11}$$

where we have used

$$E^{(k)} = \frac{L^{(k-1)}}{\varepsilon}, \qquad E^{(k)} = -\frac{1}{2}L^{(k)} + \mathcal{O}(\varepsilon) , \tag{C.12}$$

and the dependence on ε and $\bar{\gamma}_\varepsilon$ has been omitted. Finally, the full $\mathcal{O}(\varepsilon)$ correction to the action (4.8) is obtained by multiplying the saddle point fluctuations of eq. (C.11) by the first order correction to the action at the saddle point in eq. (C.9).

C.2 Cancellation of the singularities

The terms of the $\mathcal{O}(\varepsilon)$-correction to the gluon density S_1 present various singularities deriving from those of the function $L'_{\varepsilon,b}(\gamma)$ written in eq. (4.11): (1) poles at $\bar{\gamma}_\varepsilon = 0$, i.e., at $\bar{\alpha}_s(t) = 0$; (2) poles at $\chi_\varepsilon(\bar{\gamma}_\varepsilon) = b\omega/\varepsilon$, i.e., at $\bar{\alpha}_s(t) = \varepsilon/b$; (3) poles at $\varepsilon = 0$ because of the $b\bar{\alpha}_s/\varepsilon$-expansion in the denominator of $L^{(n)}_{\varepsilon,b}$.

In order to better understand the role of these singularities, we rewrite S_1 in terms of the inverse of $L'_{\varepsilon,b}$

$$\lambda(\gamma) \equiv \frac{1}{L'_{\varepsilon,b}(\gamma)} = \frac{\chi_\varepsilon(\gamma) - \frac{b\omega}{\varepsilon}}{-\chi'_\varepsilon(\gamma)} \tag{C.13}$$

and of the eigenvalue functions χ_ε and its derivatives. By computing the derivatives of λ

$$\lambda' = -\frac{L''}{(L')^2} = 1 - \frac{\chi''_\varepsilon}{\chi'_\varepsilon}\lambda, \quad \lambda'' = 2\frac{(L'')^2}{(L')^3} - \frac{L'''}{(L')^2} = -\frac{\chi''_\varepsilon}{\chi'_\varepsilon} + \left[\left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)^2 - \left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)'\right]\lambda \tag{C.14}$$

we deduce

$$\frac{(L'')^2}{(L')^3} = \frac{1}{\lambda} - 2\frac{\chi''_\varepsilon}{\chi'_\varepsilon} + \lambda\left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)^2, \quad \frac{L'''}{(L')^2} = \frac{2}{\lambda} - 3\frac{\chi''_\varepsilon}{\chi'_\varepsilon} + \lambda\left[\left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)' + \left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)^2\right], \tag{C.15}$$

so as to obtain

$$S_1 = \frac{1}{24}\frac{\chi''_\varepsilon}{\chi'_\varepsilon} + \left[\frac{1}{8}\left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)' - \frac{1}{12}\left(\frac{\chi''_\varepsilon}{\chi'_\varepsilon}\right)^2\right]\lambda, \tag{C.16}$$

which agrees with eq. (4.12) after replacing the expression (C.13) of λ and evaluating at $\gamma = \bar{\gamma}_\varepsilon$.

D. Expression of γ_{qg} by quadratures

In this appendix we want to show how it is possible to give an expression for the NLx resummation of $\gamma_{qg}^{(\overline{\mathrm{MS}})}$ in terms of the universal functions $R(\bar{\alpha}_s/\omega, \varepsilon)$ and $\mathcal{H}(\varepsilon)$ up to quadratures.

We start from eq. (6.17) and notice that, besides the overall factor $\alpha_s(t)$, all the remaining dependence on $\alpha_s(t)$ occurs through the ratio $a \equiv a(t) \equiv \bar{\alpha}_s(t)/\omega$. In fact, also the operator $\hat{D}$ defined in eq. (6.16) can be written as $\hat{D} = a\partial_a$. Let us then introduce the operator

$$\hat{O} \equiv \gamma_0[1 + \hat{D}]^{-1} = \gamma_0(a)[\partial_a \circ a]^{-1} \tag{D.1}$$

and, by using the expansions (6.13,6.14), rewrite eq. (6.17) in the form

$$\gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})} = \alpha_{\mathrm{s}} \sum_{m=0}^{\infty} \mathcal{H}_m \hat{O}^m \sum_{n=0}^{\infty} \hat{O}^n R_n(a)\,. \tag{D.2}$$

The first step is to derive an expression for arbitrary powers of $\hat{O}$. We solve this problem by considering the resolvent of $\hat{O}$

$$Z_{\hat{O}}(\lambda) \equiv [1-\lambda\hat{O}]^{-1} = \sum_{k=0}^{\infty} \lambda^k \hat{O}^k\,, \tag{D.3}$$

where λ is a complex parameter. The resolvent is the generating function of the powers of $\hat{O}$, in the sense that

$$\hat{O}^n = \frac{1}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} Z_{\hat{O}}(\lambda)\bigg|_{\lambda=0}\,. \tag{D.4}$$

An expression for $Z_{\hat{O}}(\lambda)$ is obtained by solving the equation

$$y(a) = [Z_{\hat{O}}(\lambda)x](a) \qquad \Longleftrightarrow \qquad x = [1-\lambda\hat{O}]y = y - \lambda\gamma_0[1+\hat{D}]^{-1}y\,, \tag{D.5}$$

where $y(a)$ is the unknown function to be determined in terms of $x(a)$. By introducing the function $w(a)$ such that $y = [1+\hat{D}]w = \partial_a(aw)$, we obtain a first order differential equation for the function $aw(a)$ which can be easily solved. The general solution of eq. (D.5) is

$$[Z_{\hat{O}}(\lambda)x](a) = \partial_a \int^a \mathrm{d}a'\ \exp\left[\lambda \int_{a'}^{a} \frac{\mathrm{d}a''}{a''}\,\gamma_0(a'')\right] x(a')\,. \tag{D.6}$$

The lower bound of the a'-integral is 0. It can be determined by comparing the expression of $\hat{O}$ we obtain from Eqs. (D.4) and (D.6)

$$[\hat{O}x](a) = \frac{\mathrm{d}}{\mathrm{d}\lambda}[Z_{\hat{O}}(\lambda)x](a)\bigg|_{\lambda=0} = \partial_a \int^a \mathrm{d}a'\ x(a') \int_{a'}^{a} \frac{\mathrm{d}a''}{a''}\,\gamma_0(a'') = \frac{\gamma_0(a)}{a}\int^a \mathrm{d}a'\ x(a') \tag{D.7}$$

with the integral representation of the operator $\hat{O}$ that can be obtained by the series expansion of $x(a) = \sum_{n=0}^{\infty} x_n a^n$:

$$[\hat{O}x](a) = \left[\gamma_0 \frac{1}{1+\hat{D}}x\right](a) = \gamma_0(a)\sum_{n=0}^{\infty}\frac{x_n a^n}{1+n} = \frac{\gamma_0(a)}{a}\int_0^a \mathrm{d}a'\ x(a')\,. \tag{D.8}$$

The second step is to perform the sum in n in eq. (D.2). By using again Eqs. (D.4) and (D.6) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\hat{O}^n R_n(a) &= \partial_a \int_0^a \mathrm{d}a'\ \left[\int_{a'}^{a}\frac{\mathrm{d}a''}{a''}\gamma_0(a'')\right]^n \frac{R_n(a')}{n!} \\ &= \partial_a \int_0^a \mathrm{d}a'\ R_B\Big(a', -\int_{a'}^{a}\frac{\mathrm{d}a''}{a''}\gamma_0(a'')\Big) \equiv \hat{R}(a)\,,\end{aligned} \tag{D.9}$$

where we have noticed that the sum inside the integral is just the Borel transform of the function $\mathcal{H}^{\mathrm{tran}}(\varepsilon)R(a,\varepsilon)$ in the $(-\varepsilon)$-variable.

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The third step is to perform the sum in m in eq. (D.2). Here we exploit the particular structure of the coefficients $\mathcal{H}_m$ of being a linear combination of powers, since $\mathcal{H}^{\mathrm{rat}}(\varepsilon)$ (cf. eq. (6.11)) is a linear combination of simple fractions:

$$\mathcal{H}_m = \frac{T_R}{4\pi}\left[2^m + \frac{1}{3}\left(\frac{2}{3}\right)^m\right], \tag{D.10}$$

so that, by using the second equality of eq. (D.3), we obtain just a linear combination of resolvents with parameters $\lambda = 2$ and $\lambda = 2/3$:

$$\sum_{m=0}^{\infty} \mathcal{H}_m \hat{O}^m = \frac{T_R}{4\pi}\sum_{m=0}^{\infty}\left[2^m\hat{O}^m + \frac{1}{3}\left(\frac{2}{3}\right)^m \hat{O}^m\right] = \frac{T_R}{4\pi}\left[Z_{\hat{O}}(2) + \frac{1}{3}Z_{\hat{O}}\Big(\frac{2}{3}\Big)\right]. \tag{D.11}$$

Finally, by inserting Eqs. (D.9) and (D.11) in eq. (D.2) yields

$$\gamma_{\mathsf{qg}}^{(\overline{\mathrm{MS}})} = \alpha_{\mathrm{s}}\frac{T_R}{4\pi}\partial_a\int_0^a \mathrm{d}a'\ \left[\exp\left(2\int_{a'}^a\frac{\mathrm{d}a''}{a''}\,\gamma_0(a'')\right) + \frac{1}{3}\exp\left(\frac{2}{3}\int_{a'}^a\frac{\mathrm{d}a''}{a''}\,\gamma_0(a'')\right)\right]\hat{R}(a')\,. \tag{D.12}$$

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